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E X E R C I S E S

F O R

Young Proficients

I N T H E

M A T H E M A T I C K S.

Containing,

I. A large Variety of Algebraical Problems with their Solutions.

II. A choice Number of Geometrical Problems with their Solutions both Algebraical and Geometrical.

III. THE Theory of Gunnery, independent of the Conic Sections.

IV. A new and very comprehensive Method for finding the Roots of Equations in Numbers.

V. A short Account of the Nature and first Principles of Fluxions.

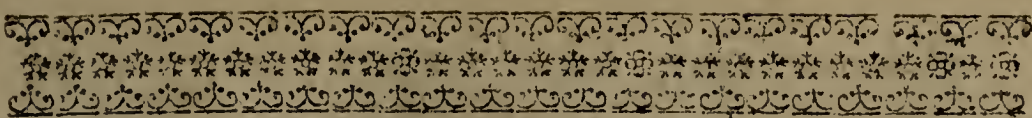
VI. THE Valuation of Annuities for single and joint Lives, with a Set of new Tables, far more extensive than any extant.

By THOMAS SIMPSON, F. R. S.

L O N D O N:

Printed for J. Nourse, opposite *Katharine-Street* in
the *Strand*. M.DCC.LII.





T O

JOHN BACON,

Of *Newtoncap*, Esq; F. R. S.

S I R,



H E N Gentlemen of your Station and Figure become the Patrons of Science it is a Benefit to the Publick, their Expectations of farther Improvements having then the best Foundation. And All who have the Pleasure of your Acquaintance, and know your Attachment to polite and useful Learning, in which a Knowledge of the Mathematicks may be justly included, will be sensible of my Happiness in being thus permitted to address You.

Believe

Believe me, Sir, whatever may be
the Fate of these Sheets, I shall, at
all Times, consider this Use of your
Name as a singular Honour to,

SIR,

Your most obedient, and most

Humble Servant,

T. SIMPSON.

*Woolwich, May 1,
1752.*



THE P R E F A C E.



THE ensuing Work, or at least the greatest Part of it, was originally composed for my own Use in the ROYAL ACADEMY: And it is upon a Presumption that It may also be of Service to Others, especially Those employed in a like Publick Way of Teaching, that It now appears in the World.

The Work itself consists of six distinct Parts, or Tracts; each of which I shall here give some Account of.

The first Part contains a Number of Algebraical Problems, with their Solutions; design'd as proper Exercises for Young Beginners. In the Course of these Problems and Solutions (whereof the greater Part will appear to be new) the Art of managing Equations, and the various Methods of Substitution are taught and illustrated.

ii The P R E F A C E.

The second Part comprehends a Variety of Geometrical Problems with their Solutions, both by Algebra and also independent of it, from Principles purely Geometrical. In this Part the Learner will find a large Field to exercise his Industry in: He will moreover have the Opportunity of comparing the two Methods of Solution together, and from thence observing, that sometimes the One has the Advantage, and sometimes the Other; that, in some Cases They both proceed upon the very same Properties, and in others, upon quite different Ones: And it may be further remarked from hence (which will be of some Use to know) that, when Quantities are given in Magnitude only, the Algebraic Method generally claims the Preference, in Point of Ease and Expedition, at least; whereas the Advantage is almost always on the Side of the Geometrical Effect, when the Positions of Points and Lines, and the Quantities of Angles are given.

There is, however, one Particular, or Two, in this Part, that may be thought to stand in need of some Apology.

In the first Place, the frequent Use of Symbols, common to the Algebraic Notation, may, perhaps, be look'd upon as repugnant to the
Rigour

Rigour and Strictness of Geometry. But it is not the Use of Symbols (which Some, more scrupulous than discerning, have condemned) but the Ideas annexed to them, that render the Consideration Geometrical, or Ungeometrical. In pure Geometry regard is always had to the absolute Quantity of some One of the three Kinds of Extension, abstractedly considered; and, whatever Symbols are used Here, are to be considered as expressive of the Quantities themselves, and not as any Measures, or numerical Values of them. Thus by $A \times B$, taken in a geometrical Sense, we have an Idea, not of the Product of two Numbers (as in the Algebraic Notation), but of a real, rectangular, Space comprehended under two Right-lines, represented by A and B, and two Others equal to them. So, likewise, $\frac{B \times C}{A}$ is not to be understood here in the Light of an Algebraic Fraction, but as a Right-line, which is Fourth Proportional to three other Right-lines, represented by A, B, and C.—These Distinctions are absolutely necessary to Those who would have an accurate Idea of the Subject.

The second Particular, above hinted at, relates to the Quotations; wherein I have referred to my own Elements of Geometry, and
not

iv The P R E F A C E.

not to those of Euclid, so universally known and established. But for this there were two Reasons: First, those Persons, for whose Instruction these Sheets are, in a more particular manner, designed, are taught the First Principles of Geometry from other Elements than those of Euclid; and, secondly, a Number of Propositions are used Here that are only to be met with in modern Authors.

In the third Part the Theory of Gunnery, or the Motion of Projectiles, is considered, exclusive of the Conic Sections; and the practical Solutions of the several Cases depending on the Theory (as well Those where the Object is elevated or depressed as where it is situate in the Plain of the Horizon) are given, at large, by Plane Trigonometry.

The fourth Part exhibits a new, and very comprehensive Method for extracting the Roots of algebraical Equations; whereby the Number sought may be determined, to any proposed Degree of Exactness, without the Trouble of repeating the Operation, as in the common Way, by Converging Series's.

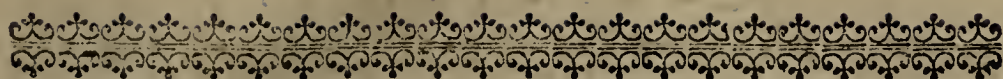
*The fifth Part gives some Account of the Nature of Fluxions, together with the Investigation of the Fundamental Rules; and may be of Use, not only to Beginners, but also to
Such,*

Such, who, though tolerably well versed in the Practice and Application of Fluxions, have nevertheless but an imperfect Idea of the First Principles of this difficult Branch of Science.

The sixth, and last, Part, is concerned about the Valuation of Annuities, on single, and joint Lives; wherein, besides a new Set of Tables, far more comprehensive than any yet published, are given the Solutions of upwards of Forty different Problems, on the most important and intricate Cases of the Subject: Many of which are quite new, and are, besides, such as actually occur in Business, being, most of them, taken from real Cases, proposed to the Author's Consideration, by Gentlemen in the Law, and Others.

This, sixth, Part, which the Reader will perceive is upon a different Plan from the five preceding Ones, was designed as a Supplement to my Doctrine of Annuities and Reversions, printed in 1742; but, being thought too small to publish alone, it is inserted Here. This, if a Fault, will not, perhaps, be look'd upon as inexcusable: Though, as to the Performance itself, I do not in the least doubt but that it will be depreciated by Some, on Account of the Observations whereon the Calculations are grounded. I am sensible that there neither is, nor
can

can be, a Table of Observations on the Degrees of Mortality of Mankind, but what may be objected to; and that those Persons who make a very despicable Figure, when They come to Calculations, seldom fail of Displaying their Talents, and being illustrious Here; where, gratifying themselves in the Liberty which the Nature of the Subject allows them, They can boldly lanch out, without having to do with Science and Demonstration. But, though the London Bills of Mortality, whereon I build, appear to me to be the best Foundation, at least, for this Place; yet I have no Inclination to enter the Lists with any of these Gentlemen. The Examples, given hereafter, are indeed wrought according to the London Bills; but the Solutions themselves are general, and may be apply'd, with equal Facility and Advantage, to any Table of Observations.



PART I.

CONTAINING

A select Number of ALGEBRAICAL PROBLEMS, with their SOLUTIONS.

DESIGN'D

As proper Exercises for young Beginners.

QUESTION I.

*WHAT Number is that, which being doubled and
16 added to the Product, the Sum shall be 188?*

Let x represent the required Number;
then $2x$ will denote the Double thereof;
and so $2x + 16 = 188$, by the Question.
Therefore $2x = 188 - 16 = 172$, by Transposition.

And $x = \frac{172}{2} = 86$, by Division.

QUESTION II.

*To find that Number, which being added to 56, the Treble
of the required Number shall be produced.*

If x be put for the Number sought,
then $3x$ will be the Treble thereof:
And therefore $3x = x + 56$, by the Question.
Hence $2x = 56$, by Transposition.

And $x = \frac{56}{2} = 28$, by Division.

B

QUESTION

QUESTION III.

The Sum of 155 l. was raised (for a certain Purpose) by three different Persons, A, B, and C; whereof B advanced 15 l. more than A; and C, 20 l. more than B: How much did each contribute?

Let x be the Number of Pounds advanced by A:
 Then $x+15$, is the Number of Pounds advanced by B,
 and $x+35$ the Number of Pounds advanced by C.
 Therefore $3x+50=155$, by the Question.
 Whence $3x=105$,
 and $x=35$.
 From which it also appears that B contributed 50 l. and C 70 l.

QUESTION IV.

A Gentleman (by Will) left 550 l. to be divided among four Servants A, B, C, and D; whereof B was to have twice as much as A; C as much as A and B; and D as much as C and B. How much had Each?

Let x be the Number of A's Pounds.
 Then $2x$ is the Number of B's Pounds,
 also $3x$ is the Number of C's Pounds,
 and $5x$ the Number of D's Pounds:
 Therefore $11x=550$, by Question.
 And, consequently, $x = \frac{550}{11} = 50$.

From which the rest of the Shares are easily determined.

QUESTION V.

'Tis required to divide the Number 92 into four such Parts; that the-First may exceed the Second by 10, the Third by 18, and the Fourth by 24.

Let x be the first Part.

Then $\left\{ \begin{array}{l} x-10 \\ x-18 \\ x-24 \end{array} \right\}$ will be the other Parts.

And

And $4x - 52 = 92$, by the Question.

Hence $4x = 144$;

and $x = \frac{144}{4} = 36$.

QUESTION VI.

A certain Sum of Money was shared among five Persons, A, B, C, D, and E; whereof B received 10 l. less than A; C 16 l. more than B; D 5 l. less than C; and E 15 l. more than D: Moreover it appeared that E received as much as both A and B. What was the whole Sum shared, and how much did Each receive?

Let x be the Share of A.

Then $\begin{cases} x-10 \\ x+6 \\ x+1 \\ x+16 \end{cases}$ will be $\begin{cases} B, \\ C, \\ D, \\ E: \end{cases}$ that of

Therefore $x + 16 = 2x - 10$, by the Question.

Whence $26 = x$.

From which it appears that 26, 16, 32, 27, and 42 (Pounds) were the respective Shares; and that the whole Sum was 143 l.

QUESTION VII.

To find that Number, whereof the Double increased by 24, shall as much exceed 80, as the Number itself is below 100.

Let x be the required Number.

Then $2x + 24 - 80 = 100 - x$, by the Question.

Whence $2x + x = 100 - 24 + 80$,

that is $3x = 156$;

and therefore $x = \frac{156}{3} = 52$.

QUESTION VIII.

What two Numbers are those, whereof the Difference is 7 and the Sum 33.

Let x be the lesser Number,
then $x + 7$ will be the greater,
and $2x + 7 = 33$.

Therefore $2x = 33 - 7 = 26$,

and $x = \frac{26}{2} = 13 =$ the Lesser,

Whence $x + 7 = 20 =$ the Greater.

QUESTION IX.

To divide the Number 75 into two such Parts, that 3 times the greater may exceed 7 times the lesser by 15.

If x be the greater Part,
then $75 - x$ will be the Lesser } *by the Question.*
and $3x = 75 - x \times 7 + 15$
that is, $3x = 525 - 7x + 15$;
therefore $10x = 540$, and $x = 54$.
From whence the lesser Part $(75 - x)$ is found $= 21$.

QUESTION X.

A, after winning 10 Guineas of B, had as much Money as B and 6 Guineas more; and betwixt them both they had forty Guineas: What Money had Each at first?

Let x be the Guineas that A began with;
then $40 - x$, are the Guineas that B began with:
Therefore, after Play,

A had, $x + 10$ Guineas;

and B, $30 - x$, Guineas.

Whence $x + 10 = 30 - x + 6$ (*by the Question.*)

Therefore $2x = 26$.

And $x = 13$.

QUESTION

QUESTION XI.

The Sum of 500 l. was divided among four Persons, so that the First and Second, between them, had 280 l; the First and Third 260 l; and the First and Fourth 220 l. How many Pounds had Each?

If x be the Number of Pounds the First had,

then the $\left\{ \begin{array}{l} 2^{\text{d}} \\ 3^{\text{d}} \\ 4^{\text{th}} \end{array} \right\}$ had $\left\{ \begin{array}{l} 280-x \\ 260-x \\ 220-x \end{array} \right\}$

The Sum of all which,

being $760-2x$, is $= 500$, by the Question.

Whence $x = \frac{760-500}{2} = 130$:

Therefore the four Shares were 130, 150, 130, and 90 Pounds, respectively.

QUESTION XII.

'Tis proposed to divide 60 into two such Parts, that the Difference between the Greater and 64, may be equal to twice the Difference between the Lesser and 38.

If x be the greater Part;

then $60-x$ will be the lesser:

Also $64-x$, will be the first mentioned Difference,

and $38-60+x$, or $x-22$, will be the Second.

Therefore $64-x = 2 \times x-22$, by the Question.

that is, $64-x = 2x-44$.

Whence $108 = 3x$, and $36 = x$.

QUESTION XIII.

After 34 Gallons had been drawn out of one of two, equal, Casks, and 80 Gallons out of the Other, there remained just twice as much Liquor in the One as in the Other: What did each Cask contain when full?

Let x be the Number of Gallons sought;

then $x-34$ will be what remained in the first Cask,

B 3

and

and $x-80$, what remained in the Second.

Hence $x-34=2 \times \overline{x-80}$, by the *Question*.

Or, $x-34=2x-160$.

Therefore $126=x$.

QUESTION XIV.

A Son, asking his Father how old he was, received the following Answer: Your Age four Years ago, says the Father, was only of mine, at that Time; but now your Age is just of mine: What was the Age of Each?

Let x be the Age of the Son,
then $3x$ will be that of the Father:
Also $x-4$ will be the Age of the Son, four Years before the Time in Question; and $3x-4$ will be the corresponding Age of the Father: which, by the *Question*, is equal to 4 times $x-4$: Hence we have this Equation
 $4x-16=3x-4$.

Therefore $x=12$, and $3x=36$; which are the two Ages required.

QUESTION XV.

What Number is that, whose $\frac{1}{3}$ exceeds its $\frac{1}{4}$ Part by 16?

Let x be the required Number; Then, its $\frac{1}{3}$ Part being $\frac{x}{3}$, and its $\frac{1}{4}$ Part $\frac{x}{4}$,

we have $\frac{x}{3} - \frac{x}{4} = 16$, by the *Question*.

Hence $4x-3x=192$, by *Reduction*;
that is, $x=192$.

QUESTION

QUESTION XVI.

In a Mixture of Wine and Cyder, one half of the whole + 25 Gallons was Wine; and, $\frac{1}{3}$ Part - 5 Gallons, Cyder. How many Gallons were there of Each?

If x be put for the Number of Gallons in the whole Mixture, the Gallons of Wine will be expressed by $\frac{x}{2} + 25$, and those of Cyder by $\frac{x}{3} - 5$: Which together being equal to the *whole*, we

therefore have $\frac{x}{2} + 25 + \frac{x}{3} - 5 = x$;

$$\text{Or, } 20 = x - \frac{x}{2} - \frac{x}{3} :$$

Hence $120 = 6x - 3x - 2x$;

Or $120 = x$. From which it appears that the Mixture consisted of 85 Gallons of Wine, and 35 of Cyder.

QUESTION XVII.

In a Lottery, consisting of 100000 Tickets, half the Number of Prizes added to $\frac{1}{3}$ of the Number of Blanks, was 35000. How many Prizes were there in the Lottery?

If x be the Number of Prizes ;
then $100000 - x$, will be the Number of Blanks.

And so, $\frac{x}{2} + \frac{100000 - x}{3} = 35000$.

Hence $3x + 200000 - 2x = 210000$;

And $x = 10000 =$ the Number sought.

QUESTION XVIII.

To the Composition of a certain Quantity of Gunpowder, $\frac{1}{2}$ of the whole Weight + 6 lb. of Saltpetre was necessary; the Sulphur used was $\frac{1}{3}$ of the whole — 5 lb. and the Charcoal $\frac{1}{4}$ of the whole — 3 lb. How many Pounds of each of the three Ingredients were there taken?

Let x be the Number of Pounds in the Whole: Then

$$\text{there were } \begin{cases} \frac{x}{2} + 6, & \text{Pounds of Saltpetre.} \\ \frac{x}{3} - 5, & \text{Pounds of Sulphur.} \\ \frac{x}{4} - 3, & \text{Pounds of Charcoal.} \end{cases}$$

And therefore $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} - 2 = x$, by the Question.

Whence $12x + 8x + 6x - 48 = 24x$:

And consequently $x = \frac{48}{2} = 24$.

Therefore there were taken 18 lb. of Saltpetre, 3 lb. of Sulphur, and 3 lb. of Charcoal.

QUESTION XIX.

A General, after having lost a Battle, found that he had only $\frac{1}{2}$ of his Army + 3600 left, fit for Action; $\frac{3}{8}$ of his Men + 600 being wounded, and the Rest, which were $\frac{1}{5}$ of the whole Army, either slain, taken Prisoners, or missing. What was the whole Number of the Army?

The Number fought being denoted by x , the Number of Men left unhurt will be $\frac{x}{2} + 3600$.

And the Number of the Wounded $\frac{x}{8} + 600$:

To which adding, $\frac{x}{5}$, the Number of the Slain and Prisoners, we have the Number of the whole Army;

Or

$$\text{Or } \frac{x}{2} + 3600 + \frac{x}{8} + 600 + \frac{x}{5} = x.$$

$$\text{Hence } 4200 \left(= x - \frac{x}{2} - \frac{x}{8} - \frac{x}{5} \right) = \frac{3x}{8} - \frac{x}{5}$$

$$\text{Therefore } 168000 (= 15x - 8x) = 7x :$$

$$\text{And } 24000 = x.$$

QUESTION XX.

A Prize of 2000 l. was divided between two Persons, A and B; whose Shares therein were in proportion as 7 to 9: What was the Share of Each?

Let x denote the Share of A;
then $2000 - x$, will be That of B.

But $x : 2000 - x :: 7 : 9$, by the Question.

From whence (as the Rectangle of the two Extremes, of any four proportional Numbers, is equal to the Rectangle of the two Means) we get this Equation,

$$9 \times x = 2000 - x \times 7;$$

$$\text{that is, } 9x = 14000 - 7x.$$

$$\text{Hence } 16x = 14000,$$

$$\text{and } x = 875.$$

Therefore the Share of A was 875 l. and that of B, 1125 l.

QUESTION XXI.

To divide 44 into two such Parts, that the Greater increased by 5, may be to the Lesser increased by 7, as 4 is to 3.

If x be the greater Part,

$44 - x$, will be the Lesser,

and $x + 5 : 51 - x :: 4 : 3$, by the Question.

Therefore (by multiplying Extremes and Means) we have

$$3x + 15 = 204 - 4x:$$

$$\text{Whence } x = \frac{204 - 15}{7} = \frac{189}{7} = 27.$$

QUESTION

QUESTION XXII.

To find two Numbers in the proportion of 1 to 2 ; so that, 12 being added to Each, the Sums shall be in proportion as 5 is to 7.

Let x be the lesser Number ;
 then $2x$, will be the Greater :
 Hence $x + 12 : 2x + 12 :: 5 : 7$, by the Question.

Therefore $2x + 12 \times 5 = x + 12 \times 7$;

that is $10x + 60 = 7x + 84$.

From which $3x = 24$; and $x = 8$.

So that the two Numbers are 8 and 16.

QUESTION XXIII.

A, at Play, first won 5 Guineas of B, and had then as much Money as B ; but B, upon winning back his own Money and 5 Guineas more, had 5 times as much Money as A : What Money had Each, at first ?

If x be the Number of A's Guineas, at first ; then it is plain, from the Question, that $x + 10$ will be the Number of B's Guineas :

Whence $x + 10 + 5 (= x - 5 \times 5) = 5x - 25$,

that is, $40 = 4x$:

Therefore $x = 10 =$ the Number of A's Guineas ;

and $x + 10 = 20 =$ the Number of B's Guineas.

QUESTION XXIV.

A Grocer, with 56 lb. of fine Tea, at 20 Shillings a Pound, would mix a coarser sort, of 14 Shillings, so as to afford the whole, together, at 18 Shillings, per Pound : What Quantity of the latter Sort must he take ?

Let x be the Number of Pounds required ;
 then 1120 is the value of finest Sort,
 and $14x$ that of the Coarsest.

Moreover,

Moreover, the Number of Pounds of both Sorts together being $56 + x$, it is evident

that $56 + x \times 18$, or $1008 + 18x$, is the Value of the whole Mixture. And therefore

$$1120 + 14x = 1008 + 18x:$$

Whence $112 = 4x$:

And consequently $28 = x$.

QUESTION XXV.

A Farmer would mix Wheat, at 4 Shillings a Bushel, with Rye, at 2 s. 6 d. a Bushel; so that the whole Mixture may consist of 90 Bushels, and be afforded at 3 s. 4 d. a Bushel: 'Tis required to find how many Bushels of each Sort must be taken.

If the Number of Bushels of Wheat be x ,

Those of Rye will be $90 - x$.

Moreover, the Value of the Wheat will be $48x$, Pence, and the Value of the Rye $90 - x \times 30$, Pence:

Whence $48x + 90 - x \times 30 = 90 \times 38$ (by the Question.)

that is, $48x + 2700 - 30x = 3420$:

Therefore $18x = 720$:

And consequently $x = \frac{720}{18} = 40$.

QUESTION XXVI.

A Workman was hired for 40 Days, at 3 s. 4 d. per Day, for every Day he worked; but with this Condition, that, for every Day he played, he was to forfeit 1 s. 4 d.: And it so happened, that, upon the whole, he had 3 l. 3 s. 4 d. to receive. The Question is, to find how many Days, of the 40, he work'd, and how many he play'd.

Let x be the Number of Days he work'd;

then $40 - x$ will be the Number of Days he play'd:

Moreover, as he was to receive 40 Pence, for every Day he work'd, and to forfeit 16 Pence for every Day he play'd,

we have $40x =$ the Number of Pence earn'd by Work,
and $40-x \times 16 =$ the Number of Pence forfeited by Play:

Whence $40x - 40 - x \times 16 = 760$, by the Question;
that is, $40x - 640 + 16x = 760$:

Therefore $56x = 1400$;

and consequently $x = 25$.

From which it is evident that he work'd 25 Days, and play'd 15.

QUESTION XXVII.

A Bill of 120 l. was paid in Guineas and Moidores, and the Number of Pieces used of both Sorts was just 100: How many were there of Each?

If x be put for the Number of Guineas;
then $100-x$ will be the Number of Moidores:

And so, $21x + 100 - x \times 27 = 120 \times 20$ } by the
Or, $21x + 2700 - 27x = 2400$ } Question.

Hence $300 = 27x - 21x = 6x$.

And consequently $x = \frac{300}{6} = 50$.

QUESTION XXVIII.

One bought 30 Pounds of Sugar, of two different Sorts, and paid for the whole 19 Shillings; the best Sort cost 10d. per Pound, and the worst 7d. How many Pounds were there of Each?

Let x stand for the Number of Pounds of the best Sort,
and then $30-x$ will express the Pounds of the other Sort;

Therefore $x \times 10 + 30 - x \times 7 = 19 \times 12$, by the Question;
that is, $10x + 210 - 7x = 228$:

Whence $3x = 228 - 210 = 18$,

and $x = 6$. Therefore there were 6 Pounds of the best Sort, and 24 of the Worst.

QUESTION

QUESTION XXIX.

A Lady gave a Guinea, in Charity, among a Number of Poor, consisting of Men, Women, and Children: Each Man had 12d. each Woman 6d. and each Child 3d. Moreover there were twice as many Women as Men, wanting 2; and 3 times as many Children as Women, wanting 4: How many Persons were there relieved?

Let x be the Number of the Men;
then $2x-2$, will be the Number of Women,
and $6x-10$, the Number of Children.

Hence $x \times 12 + 2x - 2 \times 6 + 6x - 10 \times 3 = 21 \times 12$;

that is, $12x + 12x - 12 + 18x - 30 = 252$,

or, $42x = 294$;

Whence $x = 7$.

Therefore there were 7 Men, 12 Women, and 32 Children; in all 51 Persons.

QUESTION XXX.

To find that Number, which being divided, either into three, or four, equal Parts, the continual Product of all the Parts, in both Cases, shall be exactly the same.

Let x be the required Number;
so shall the continual Product of the three equal Parts

be $\frac{x}{3} \times \frac{x}{3} \times \frac{x}{3} = \frac{x^3}{27}$; and that of the four equal

Parts $\frac{x}{4} \times \frac{x}{4} \times \frac{x}{4} \times \frac{x}{4} = \frac{x^4}{256}$.

Whence $\frac{x^4}{256} = \frac{x^3}{27}$, by the Question.

Therefore $27x = 256$; and $x = 9\frac{13}{27}$.

QUESTION

QUESTION XXXI.

To find two Numbers, in the Proportion of 3 to 4, whose Sum is to the Sum of their Squares, as 7 to 50.

Let $3x$ denote the lesser Number:

Then $4x$ will express the greater.

And we shall have $3x + 4x : 9x^2 + 16x^2 :: 7 : 50$,

or, $7x : 25x^2 :: 7 : 50$, by the Question.

Therefore $25x^2 \times 7 = 7x \times 50$,

or, $25x^2 = 50x$;

whence $x = \frac{50}{25} = 2$: So that 6 and 8 are the two Num-

bers that answer the Question.

QUESTION XXXII.

To find two Numbers in the Proportion of 9 to 7; so that the Square of their Sum, and the Cube of their Difference, shall be equal.

If $9x$ be put for the greater Number;
then $7x$ will be the lesser;

And so $\overline{16x}^2 = \overline{2x}^3$, by the Question,
that is, $256x^2 = 8x^3$.

Hence $x = \frac{256}{8} = 32$:

Therefore 288 and 224, are the two Numbers sought.

QUESTION XXXIII.

To find two Numbers whose Difference is 4 and the Difference of their Squares 120.

Let x be the lesser Number;

then $x + 4$, will be the greater:

Also xx will be the Square of the lesser,

and $xx + 8x + 16$ that of the greater:

Whence $8x + 16 = 120$, by the Question:

Therefore $8x = 104$; and $x = 13$:

So that 13 and 17 are the two Numbers that were to be found.

QUESTION

QUESTION XXXIV.

To divide 100 into two such Parts, that the Difference of their Squares may be 1000.

If x be the greater Part,
 $100-x$, will be the lesser :

Therefore $xx - 100-x]^2 = 1000$;

that is, $x^2 - 10000 + 200x - x^2 = 1000$:

Whence $200x = 11000$;

and consequently $x = \frac{11000}{200} = 55$.

QUESTION XXXV.

To divide 100 into two Parts, so that the Square of their Difference may exceed the Square of twice the lesser Part by 2000.

The lesser Part being denoted by x ,
 the greater will be expressed by $100-x$,
 and the Difference by $100-2x$.

Therefore, by the Problem, $100-2x]^2 = 2x]^2 + 2000$,

that is, $10000 - 400x + 4xx = 4xx + 2000$,

or, $10000 - 2000 = 400x$.

Hence $x = \frac{8000}{400} = 20$; and $100-x = 80$.

QUESTION XXXVI.

A and B make a joint Stock of 500 l. by which they gain 160 l. whereof A, for his Share, had 32 l. more than B: What did each Person bring into Stock?

If x be the Number of Pounds advanced by A ;
 then it will be, as 500 (the whole Stock) is to 160 (the whole Gain) so is x (the Stock of A) to $\frac{160x}{500}$ the

Gain of A: Whence the Gain of B being $160 - \frac{160x}{500}$:
 we

we have $\frac{16x}{50} = 160 - \frac{16x}{50} + 32$, by the Question :

Therefore $\frac{32x}{50} = 192$, or $\frac{x}{50} = 6$;

and consequently $x = 50 \times 6 = 300$.

QUESTION XXXVII.

A Sum of Money was divided between two Persons, A and B, so that the Share of A was to That of B, as 5 to 3, and exceeded $\frac{5}{9}$ of the whole Sum by 50 l. What was the Share of each Person ?

Let $5x$ express the Share of A,
and $3x$ the Share of B ;
then $8x$ will be the whole Sum,

and $\frac{40x}{9}$ will be $\frac{5}{9}$ thereof :

Therefore $5x - \frac{40x}{9} = 50$, by the Question ;

that is, $\frac{5x}{9} = 50$, or $x = 90$.

Hence 450 l. and 270 l. are the two Shares required.

QUESTION XXXVIII.

A and B began to play together with equal Sums of Money ; A first won 20 Guineas, but afterwards lost back the Half of all he then had ; and thereupon had only half as much Money as B. What Money did each begin with ?

Let x be the Number of Guineas required :

Then A, after winning 20 Guineas, had $x + 20$; the Half of which, or $\frac{x}{2} + 10$, is therefore what he had at last : And this deducted from $2x$ (the whole Sum betwixt Both) leaves $2x - \frac{x}{2} - 10 =$ what B had at last.

Therefore

Therefore $2x - \frac{x}{2} - 10 = x + 20$;

whence $3x - 20 = 2x + 40$,
and consequently $x = 60$.

QUESTION XXXIX.

A Gentleman left his whole Estate among his four Sons ; whereof the Eldest had $\frac{1}{2}$ wanting 800 l. the Second $\frac{1}{4}$ and 120 l. over ; the Third had half as much as the Eldest ; and the Youngest $\frac{2}{3}$ of what the Second had. What was the whole Estate ? and how much had Each ?

Let x be the whole Estate ; then

The First had $\frac{x}{2} - 800$.

The Second $\frac{x}{4} + 120$.

The Third $\frac{x}{4} - 400$.

The Fourth $\frac{2x}{12} + 80$.

The Sum of all which is, consequently, equal to the Whole,

Or, $\frac{x}{2} + \frac{x}{4} + \frac{x}{4} + \frac{x}{6} - 1000 = x$.

But it is plain (without Reduction)

that $\frac{x}{2} + \frac{x}{4} + \frac{x}{4} = x$.

Hence our Equation becomes

$\frac{x}{6} - 1000 = 0$, or $\frac{x}{6} = 1000$.

Therefore the whole Estate was 6000 l. whereof the eldest Son had 2200 l. the Second 1620 l. the Third 1100 l. and the Youngest 1080 l.

QUESTION XL.

One being ask'd his Age, reply'd; If $\frac{2}{5}$ of my Years be multiply'd by 3, and $\frac{1}{3}$ of them be added to the Product, the Amount will be 115. What was his Age?

If x be the required Number of Years,

then $\frac{2x}{5} \times 3 + \frac{x}{3} = 115$, by the Question;

that is, $\frac{6x}{5} + \frac{x}{3} = 115$.

Therefore $18x + 5x = 15 \times 115$,

or $23x = 15 \times 115$:

Consequently $x = \frac{15 \times 115}{23} = 15 \times 5 = 75$.

QUESTION XLI.

A Person being ask'd the Hour of the Day, answered thus: If $\frac{3}{8}$ of the Number of Hours remaining till Midnight be multiply'd by 4, the Product will as much exceed 12 Hours, as Half the present Hour from Noon is below 4: What was the Hour after Noon?

Let x be the required Hour;

then $12 - x$, will be the Hours till Midnight,

and $\frac{36 - 3x}{8} \times 4 - 12 = 4 - \frac{x}{2}$, by the Question.

That is, $\frac{36 - 3x}{2} - 12 = 4 - \frac{x}{2}$,

whence $36 - 3x - 24 = 8 - x$;

and consequently $x = 2$.

QUESTION

QUESTION XLII.

A Market-woman bought in a certain Number of Eggs, at the Rate of 5 for two Pence; one half of which she sold out again at 2 a Penny, and the remaining Half at 3 a Penny; and cleared 4 Pence, by so doing: What Number of Eggs had she?

Let $2x$ be the Number sought;
then, by the Question,

$5 : 2 :: 2x : \frac{4x}{5}$ the Number of Pence the Eggs cost:

But the Number of Pence They were sold for, again,

is $\frac{x}{2} + \frac{x}{3}$: Therefore we have this Equation,

$$\text{viz. } \frac{x}{2} + \frac{x}{3} - \frac{4x}{5} = 4:$$

From whence $15x + 10x - 24x = 120$, or $x = 120$.

QUESTION XLIII.

A certain Sum of Money, put out at Interest, amounts, in 8 Months, to 297 l. 12 s. And, in 15 Months its Amount (computed according to simple Interest) is 306 l. What is that Sum? And what the Rate of Interest?

Let x be the Number of Pounds in the required Sum:
Then, the Interest thereof for 8 Months being $297, 6 - x$,
and for 15 Months $306 - x$, we have,

as $8 : 15 :: 297, 6 - x : 306 - x$:

Whence, by multiplying Extremes and Means,
we get $2448 - 8x = 4464 - 15x$.

Therefore $7x = 2016$,

and consequently $x = 288$ l. the Sum required.

For the Rate of Interest, it will be,

as $288 \text{ l.} \times 15 : 100 \text{ l.} \times 12 :: 18 \text{ l.}$ (the Interest of 288 l. for 15 Months) to 5 l. the required Interest of 100 l. for 12 Months.

QUESTION XLIV.

A Waterman finds by Experience, that he can, with the Advantage of a common Tide, row from London to Greenwich, which is 5 Miles, in 3 Quarters of an Hour; and that, to return to London, against an equal Tide, though he rows back along-shore, where the Stream is only half as strong as in the Middle, takes him a full Hour and Half. 'Tis required to find, from hence, at what Rate, per Hour, the Tide runs in the Middle where it is strongest.

In the first Place, it will be

qr. qr. m m.

$3 : 4 :: 5 : 6\frac{2}{3} = \text{Dist. row'd per Hour with the Tide,}$

$6 : 4 :: 5 : 3\frac{1}{3} = \text{Dist. row'd per Hour against Tide.}$

If now the former of these two Distances ($6\frac{2}{3}$) be put $=a$, and the latter ($3\frac{1}{3}$) $=b$; and x be assumed to express the required Distance run, per Hour, by the Stream in the Middle of the River; then $a-x$ will be the *real* Effect of his Rowing, per Hour, in going from London, the

Motion of the Tide being deducted; and $b+\frac{x}{2}$ will be

the like Effect in his Return:

And so, these two Quantities being equal to each other,

we have $b+\frac{x}{2}=a-x$:

Whence $2b+x=2a-2x$;

and consequently $x=\frac{2a-2b}{3}=2\frac{2}{9}$.

QUESTION XLV.

To divide 36 (a) into 3 such Parts, that $\frac{1}{2}$ of the First, $\frac{1}{3}$ of the Second, and $\frac{1}{4}$ of the Third, may be equal to each other.

If x be put for the first Part,

then is $\frac{x}{2}=\frac{1}{3}$ of the second Part, (by the Question.)

And

And so $\frac{3x}{2} =$ second Part.

Moreover $\frac{x}{2}$ being $= \frac{1}{4}$ of the third Part,

therefore $\frac{4x}{2}$ (or $2x$) = third Part.

Hence $x + \frac{3x}{2} + 2x = a$,

and $2x + 3x + 4x = 2a$.

Consequently $x = \frac{2a}{9} = \frac{2 \times 36}{9} = 2 \times 4 = 8$.

From which the second Part $\left(\frac{3x}{2} \right)$

appears to be $= 12$, and the Third $(2x) = 16$.

QUESTION XLVI.

To divide the Number 90 into 4 such Parts, that, if the first be increased by 5, the second diminished by 4, the third multiply'd by 3, and the fourth divided by 2, the Result, in each Case, shall be exactly the same.

Let x be the fourth, or last, Part:

Then, three times the third Part being $= \frac{x}{2}$,

the third Part will be $\frac{x}{6}$.

Moreover, the second Part $- 4$ being, also, $= \frac{x}{2}$,

the second Part will be $\frac{x}{2} + 4$:

And, the first Part $+ 5$ being $= \frac{x}{2}$,

the first Part, alone, will be $\frac{x}{2} - 5$.

And, by adding all the Parts thus found together,

we have $x + \frac{x}{6} + \frac{x}{2} + 4 + \frac{x}{2} - 5 = 90$;

C 3

that

that is, $2x + \frac{x}{6} - 1 = 90$.

Whence $13x = 91 \times 6$; and $x = 42$.

Therefore the four required Parts are 16, 25, 7, and 24, respectively.

QUESTION XLVII.

Two Workmen A and B were employed together for 50 Days, at 5 shillings per Day, Each; during which Time A, by spending only Sixpence a Day less than B, had saved twice as much as B, besides the expence of 2 Days over: What did each Person expend a Day?

Let x be the Pence A spent per Day;
then $60 - x$, will be what he saved per Day,
and $54 - x$, what B saved.
Therefore $3000 - 50x$ are A's whole savings,
and $2700 - 50x$ those of B.

Hence $3000 - 50x = 2 \times 2700 - 50x + 2x$;

Or, $3000 - 50x = 5400 - 98x$;

From which $48x = 2400$, and $x = 50$.

QUESTION XLVIII.

Two Persons, A and B, have both the same Income; A lays by $\frac{1}{5}$ of his; but B, by spending 60 l. per Ann. more than A, at the End of three Years finds himself 100 l. in debt. What did Each receive, and expend, per Annum?

Let x be the yearly Income of Each;
then $\frac{4x}{5}$ is the Sum expended by A, per Ann.
and $\frac{4x}{5} + 60$, That expended by B.

Therefore $\frac{4x}{5} + 60 - x$, is what B runs in Debt.

Consequently $\frac{4x}{5} + 60 - x \times 3 = 100$,

$$\text{or } \frac{12x}{5} + 180 - 3x = 100;$$

$$\text{that is, } 180 - \frac{3x}{5} = 100.$$

$$\text{Whence } 900 - 3x = 500,$$

$$\text{and } x = \frac{400}{3} = 133\text{ l. } 6\text{ s. } 8\text{ d.}$$

Therefore A expended 106 l. 13 s. 4 d. and B 166 l. 13 s. 4 d. *per Annum*.

QUESTION XLIX.

A Grazier bought in as many Sheep, of different Sorts, as cost him 33 l. 7 s. 6 d. For the first Sort, which were $\frac{1}{3}$ of the whole, he paid 9 s. 6 d. a-piece; for the second Sort, which were $\frac{1}{4}$ of the whole, he paid 11 s. each; and for the rest, 12 s. 6 d. each: What Number of Sheep did he buy in all?

If x be the whole Number of Sheep;

then, the Number of the first Sort being $\frac{x}{3}$, and of the

second Sort $\frac{x}{4}$, the Number of the remaining Sort (at 12 s.

6 d. each) must be $x - \frac{x}{3} - \frac{x}{4} = \frac{12x - 4x - 3x}{12} = \frac{5x}{12}$.

Whence, by the conditions of the Problem, we have

$$\frac{x}{3} \times 19 + \frac{x}{4} \times 22 + \frac{5x}{12} \times 25 = 1335;$$

$$\text{that is, } \frac{19x}{3} + \frac{22x}{4} + \frac{125x}{12} = 1335.$$

Let each Term of this Equation be now multiply'd by 12, and it will become

$$76x + 66x + 125x = 16020,$$

$$\text{or, } 267x = 16020.$$

Therefore $x = 60$.

QUESTION L.

A Draper, of a Piece of Cloth, standing him in 3 s. 2 d. per Yard, sold $\frac{1}{3}$ Part, at 4 s. per Yard; $\frac{1}{4}$ at 3 s. 8 d. per Yard; $\frac{1}{5}$ at 3 s. 6 d. per Yard; and the Remnant at 3 s. 4 d. a Yard: And his Gain upon the Whole, was 15 s. 2 d. How many yards did the Piece contain?

If the Number sought be denoted by x ;
then the Number of yards in the Remnant

will be $x - \frac{x}{3} - \frac{x}{4} - \frac{x}{5} = \frac{60x - 20x - 15x - 12x}{60} = \frac{13x}{60}$.

Therefore, by the Question, we have

$$\frac{x}{3} \times 48 + \frac{x}{4} \times 44 + \frac{x}{5} \times 42 + \frac{13x}{60} \times 40 - 38x = 182,$$

$$\text{Or, } 16x + 11x + \frac{42x}{5} + \frac{26x}{3} - 38x = 182,$$

$$\text{that is } \frac{42x}{5} + \frac{26x}{3} - 11x = 182:$$

$$\text{Whence } 126x + 130x - 165x = 2730;$$

$$\text{And therefore } x = \frac{2730}{91} = 30.$$

QUESTION LI.

A Distiller proposes to mix Foreign Brandy, standing him in 8 Shillings a Gallon, with British Spirits of 3 Shillings per Gallon, in such Proportion that he may gain 30 per Cent by selling out the Compound at 9 s. a Gallon. What is that Proportion?

Suppose, that, with a Gallons of Brandy, he mixes x Gallons of Spirits; then, the Brandy, standing him in $8a$ (Shillings) and the Spirits in $3x$ (Shillings), the true value of the whole Mixture will be $8a + 3x$:

But the Value of $a + x$ Gallons, at 9 Shillings per Gallon, is $9a + 9x$: Therefore, by laying out $8a + 3x$, he gains $a + 6x$: And so

we have $8a + 3x : a + 6x :: 100 : 30$, by the Question.

Consequently

Consequently $100a + 600x = 240a + 90x$;

whence $510x = 140a$,

$$\text{and } x = \frac{14a}{51}.$$

From which it appears that, to every 51 Gallons of Brandy, there must be taken 14 Gallons of Spirits.

QUESTION LII.

To find two Numbers in the Proportion of 4 to 5, from which two other (required) Numbers, in the proportion of 6 to 7 being, respectively, deducted, the Remainders shall be in the proportion of 2 to 3, and their Sum equal to 20.

Let $4x$ and $5x$ be the 2 first Numbers, and $6y$ and $7y$, the other 2 Numbers.

Then $4x - 6y : 5x - 7y :: 2 : 3$ } *by the Question.*

And $9x - 13y = 20$

From which Proportion, by multiplying Extremes and Means, we have $12x - 18y = 10x - 14y$, and therefore $x = 2y$; which substituted in the above Equation gives $18y - 13y = 20$;

whence $y = \frac{20}{5} = 4$; and $x (=2y) = 8$.

Therefore the 2 first Numbers are 32 and 40; and the other Two, 24 and 28.

QUESTION LIII.

A Farmer sold, at one time, 30 Bushels of Wheat and 40 of Barley, and for the whole received 13 l. 10 s; and, at another time, he sold 50 Bushels of Wheat and 30 of Barley, at the same Prices as before, and for the whole received 17 l. The Question is, to find what each Sort of Grain was sold at per Bushel.

Let x and y express the Numbers of Shillings, respectively, that the Wheat and Barley were sold at per Bushel; and then, from the Conditions of the Question, we shall have the two following Equations, viz.

$$30x$$

$$30x + 40y = 270,$$

$$50x + 30y = 340.$$

From 4 times the Second of which Equations let 3 times the First be subtracted, and there will remain

$$110x = 550.$$

Therefore $x = \frac{55}{11} = 5:$

And $y \left(= \frac{27 - 3x}{4} \right) = 3.$

QUESTION LIV.

A Farmer, with 28 Bushels of Barley, at 2 s. 4 d. per Bushel, would mix Rye, at 3 s. per Bushel, and Wheat, at 4 s. per Bushel; so that the whole Mixture may consist of 100 Bushels, and be worth 3 s. 4 d. a Bushel: How many Bushels of Rye, and how many of Wheat must be mingled with the Barley?

Let x be the Number of Bushels of Rye, and y those of the Wheat: Then, the value of the Barley being 784 (Pence), of the Rye $36x$ (Pence), and of the Wheat $48y$ (Pence), we have

$$\left. \begin{array}{l} 784 + 36x + 48y = 4000 \\ \text{and } 28 + x + y = 100 \end{array} \right\} \text{by the Question.}$$

From the first of which Equations, take 36 times the second, and there results,

$$784 - 36 \times 28 + 12y = 400,$$

$$\text{that is, } -224 + 12y = 400.$$

$$\text{Therefore } 12y = 624,$$

$$\text{and consequently } y = \frac{624}{12} = 52.$$

$$\text{Whence } x (=100 - 28 - y) = 20.$$

QUESTION

QUESTION LV.

A and B, working together on the same Work, can earn 40 Shillings in 6 Days; A and C together can earn 54 Shillings in 9 Days; and B and C, 80 Shillings in 15 Days: 'Tis required to find what each Person, alone, can earn per Day.

Let x , y and z express the Numbers of Shillings in the three required Values, respectively.

$$\text{Then } \begin{cases} 6x + 6y = 40 \\ 9x + 9z = 54 \\ 15y + 15z = 80 \end{cases} \text{ by the Question.}$$

$$\text{And } \begin{cases} x + y = 6\frac{2}{3} \\ x + z = 6 \\ y + z = 5\frac{1}{3} \end{cases} \text{ by Division.}$$

Hence $y - z = \frac{2}{3}$, by subtracting the 2d Equation from 1st.

And $2y = 6$, by adding the two last.

Consequently $y = 3$: From which

we have $x (= 6\frac{2}{3} - y) = 3\frac{2}{3} = 3s. 8d.$ and $z (= 5\frac{1}{3} - y) = 2\frac{1}{3} = 2s. 4d.$

QUESTION LVI.

To find three Numbers, so that $\frac{1}{2}$ the First, $\frac{1}{3}$ of the Second, and $\frac{1}{4}$ of the Third, shall together be equal to 62; also $\frac{1}{3}$ of the First, $\frac{1}{4}$ of the Second, and $\frac{1}{5}$ of the Third, equal to 47; and, lastly, $\frac{1}{4}$ of the First, $\frac{1}{5}$ of the Second, and $\frac{1}{6}$ of the Third, equal to 38.

Put $a=62$, $b=47$, and $c=38$; and let the three required Numbers be denoted by x , y and z , respectively; then the Conditions of the Problem will be expressed in the three following Equations, viz.

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = a,$$

$$\frac{x}{3} + \frac{y}{4} + \frac{z}{5} = b,$$

$$\frac{x}{4} + \frac{y}{5} + \frac{z}{6} = c.$$

Which,

Which, clear'd of Fractions, become

$$12x + 8y + 6z = 24a$$

$$20x + 15y + 12z = 60b$$

$$30x + 24y + 20z = 120c.$$

Now (in order to exterminate z) let the Second of these Equations be taken from the Double of the First, and also the Treble of the Third from the Quintuple of the Second; and there results

$$4x + y = 48a - 60b; \text{ and}$$

$$10x + 3y = 300b - 360c:$$

Whence, by deducting the second of These from the Treble of the Former, and dividing by 2, there comes out

$$x = 72a - 240b + 180c = 24.$$

From which $y (= 48a - 60b - 4x)$ is also found $= 60$,

and $z (= a - \frac{1}{2}x - \frac{1}{3}y \times 4) = 120$.

QUESTION LVII.

To divide the Number 90 (a) into three Parts, so that, the Double of the first Part + 40 (b); the Treble of the Second + 20 (c); and the Quadruple of the Third + 10 (d), may be all equal to one another.

Let x , y , and z represent the three required Parts, respectively; then, from the Conditions of the Problem, we shall have

$$x + y + z = a,$$

$$2x + b = 3y + c,$$

$$2x + b = 4z + d.$$

Now, in order to exterminate y and z , let 12 times the first Equation, 4 times the Second, and 3 times the Third, be added all together; and you will have

$$26x + 12y + 12z + 7b = 12a + 12y + 12z + 4c + 3d.$$

$$\text{Therefore } 26x = 12a + 4c + 3d - 7b,$$

$$\text{and } x = \frac{12a + 4c + 3d - 7b}{26} = 35.$$

$$\text{Whence } y \left(= \frac{2x + b - c}{3} \right) = 30, \text{ and } z (= a - x - y) = 25.$$

QUESTION

QUESTION LVIII.

To find three Numbers, so that the First with half the other Two, the Second with $\frac{1}{3}$ of the other Two, and the Third with $\frac{1}{4}$ of the other Two, may be the same, and amount to 51 in each Case.

Put $a = 51$, and let x , y and z denote the three required Numbers; then, by the Question,

$$x + \frac{y+z}{2} = a,$$

$$y + \frac{x+z}{3} = a,$$

$$z + \frac{x+y}{4} = a:$$

Which, cleared of Fractions, become

$$2x + y + z = 2a,$$

$$x + 3y + z = 3a,$$

$$x + y + 4z = 4a.$$

From whence, by taking the Second from the Third, and the First from the Double of the Second, there results $-2y + 3z = a$,

$$\text{and } 5y + z = 4a.$$

And, by deducting the former of These, from the Treble of the latter, we have $17y = 11a$.

$$\text{Therefore } y = \frac{11a}{17} = 33,$$

$$z (= 4a - 5y) = \frac{13a}{17} = 39,$$

$$\text{and } x (= 3a - 3y - z) = \frac{5a}{17} = 15.$$

QUESTION

QUESTION LIX.

A certain Sum of Money was divided between three Persons, A, B and C; so that, A's Share exceeded $\frac{4}{7}$ of the Shares of B and C by 30l; also the Share of B exceeded $\frac{3}{8}$ of the Shares of A and C by 30l; and the Share of C likewise exceeded $\frac{2}{9}$ of the Shares of A and B, by 30l. The Question is, to find the Share of each Person.

Let $a = 30$; and let x , y , and z be assumed to express the three required Numbers; then by the Conditions of the Problem,

$$x - \frac{4y + 4z}{7} = a,$$

$$y - \frac{3x + 3z}{8} = a,$$

$$z - \frac{2x + 2y}{9} = a.$$

Whence, by Reduction,

$$7x - 4y - 4z = 7a,$$

$$-3x + 8y - 3z = 8a,$$

$$-2x - 2y + 9z = 9a.$$

Now, to get rid of y (which, because of the even Coefficients, is the easiest to be exterminated) let the Double of the first Equation and the Quadruple of the Third be, successively, added to the Second; by means whereof we have

$$11x - 11z = 22a,$$

$$-11x + 33z = 44a.$$

Moreover, by adding these two last Equations together, we have $22z = 66a$.

Therefore $z = 3a = 90$;

whence $x (= 2a + z) = 5a = 150$,

and $y (= a + \frac{3}{8} \times x + z) = 4a = 120$.

QUESTION

QUESTION LX.

If A and B together can perform a Piece of Work in 8 Days ; A and C together in 9 Days ; and B and C in 10 Days : How many Days will it take each Person, alone, to perform the same Work ?

Let the whole Work be represented by a , and let x , y , and z stand for the Parts thereof performed by A, B, and C in one Day, respectively.

Then will $\left\{ \begin{array}{l} 8x + 8y = a \\ 9x + 9z = a \\ 10y + 10x = a \end{array} \right\}$ by the Question.

And $\left\{ \begin{array}{l} x + y = \frac{a}{8} \\ x + z = \frac{a}{9} \\ y + z = \frac{a}{10} \end{array} \right\}$ by Division.

Whence $y - z \left(= \frac{a}{8} - \frac{a}{9} \right) = \frac{a}{72}$,

And $2y = \frac{a}{10} + \frac{a}{72} = \frac{82a}{720}$.

Consequently $y = \frac{41a}{720}$: From which $x \left(= \frac{a}{8} - y \right)$ is found $= \frac{49a}{720}$,

and $z \left(= \frac{a}{10} - y \right) = \frac{31a}{720}$.

Now, the Part of the Work (a) performed by each Person in one single Day being thus assigned, the Number of Days it will take any one of them to do the Whole, will be found by dividing the Whole by the assigned Part.

Thus, $\frac{41a}{720} \Big) a = \frac{720a}{41a} = \frac{720}{41} = 17 \frac{23}{41}$ is the Number of Days in which B, alone, can do the Whole. And, in like

like manner, the Number of Days in which A, or C, can do the Whole, appears to be

$$14\frac{34}{49}, \text{ or } 23\frac{7}{31}, \text{ respectively.}$$

QUESTION LXI.

If A, B and C can, together, finish a Piece of Work in 9 Days; A, B and D together, in 10 Days; A, C and D together, in 11 Days; and B, C and D in 12 Days: In how long Time can they all Four, together, finish It?

Here, denoting the given Numbers by $a, b, c,$ and $d,$ and putting $u, x, y,$ and $z,$ for the Parts of the whole Work (g) done by Each in one Day, respectively, we shall, by the Question, have these Equations,

$$\text{Viz. } \left\{ \begin{array}{l} a \times u + x + y = g \\ b \times u + x + z = g \\ c \times u + y + z = g \\ d \times x + y + z = g \end{array} \right\}$$

$$\text{or } \left\{ \begin{array}{l} u + x + y = \frac{g}{a} \\ u + x + z = \frac{g}{b} \\ u + y + z = \frac{g}{c} \\ x + y + z = \frac{g}{d} \end{array} \right\} \text{ by Division.}$$

The Sum of all which, divided by 3,

$$\text{gives } u + x + y + z = \frac{1}{3} \times \frac{g}{a} + \frac{g}{b} + \frac{g}{c} + \frac{g}{d}.$$

Therefore, seeing the Work done by all the Four, in one Day, is expressed by $\frac{1}{3} \times \frac{g}{a} + \frac{g}{b} + \frac{g}{c} + \frac{g}{d}$, the whole Work g , divided hereby, will consequently give

give $\frac{3g}{\frac{g}{a} + \frac{g}{b} + \frac{g}{c} + \frac{g}{d}} = \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}} =$

$\frac{3abcd}{bcd + acd + abd + abc} = 7 \frac{1797}{2289}$; for the Number of Days required.

QUESTION LXII.

To find that Number whose square Root is to its Cube Root, in the Proportion of 5 to 2.

Let x^6 express the required Number :

Then x^3 will be its square Root,
and x^2 its Cube Root.

Now $x^3 : x^2 :: 5 : 2$, by the Question.

Therefore $2x^3 = 5x^2$, or $2x = 5$;

or, lastly, $x = \frac{5}{2} = 2.5$.

Whence $x^6 = 244.140625 =$ the Number sought.

QUESTION LXIII.

*To find two Numbers in the Proportion of 3 to 5 ; where-
of the fifth Power of the First shall be to the third Power
of the Second, as 972 to 125.*

If $3x$ be put for the first Number,
then $5x$ will express the Second ; and we shall
have $\overline{3x}^5 : \overline{5x}^3 :: 972 : 125$, by the Question,
that is, $243x^5 : 125x^3 :: 972 : 125$:

Hence $243x^5 \times 125 = 125x^3 \times 972$,

Or, $243x^2 = 972$.

Therefore $x^2 = \frac{972}{243} = 4$;

and $x = \sqrt{4} = 2$: So that 6 and 10 are the two
Numbers that were to be found.

QUESTION LXIV.

To find three Numbers in the Ratio of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$; whereof the Sum of all the Squares shall be 549.

Let x denote the first Number,

then it will be $\frac{1}{2} : \frac{1}{3} :: x : \frac{2x}{3} =$ the second Number.

And $\frac{1}{2} : \frac{1}{4} :: x : \frac{x}{2} =$ the third Number.

Hence $x^2 + \left[\frac{2x}{3}\right]^2 + \left[\frac{x}{2}\right]^2 = 549$, by the Question,

that is, $x^2 + \frac{4x^2}{9} + \frac{x^2}{4} = 549$,

or, $36x^2 + 16x^2 + 9x^2 = 36 \times 549$:

From which $x^2 = \frac{36 \times 549}{61} = 36 \times 9$,

and $x = 6 \times 3 = 18$. Therefore 18, 12 and 9, are the three required Numbers.

Otherwise,

By reducing the given Fractions $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, to the same Denomination, they will appear to be in the Proportion of 6, 4 and 3. If, therefore, the first of the three Numbers sought be denoted by $6x$, the other Two will be expressed by $4x$, and $3x$, respectively: And so we shall have

$$36x^2 + 16x^2 + 9x^2 = 549.$$

Whence $x = 3$, and the Numbers sought, as before.

QUESTION LXV.

Having given the Difference of two Numbers $= 6$, and their Product $= 720$; to find the Numbers.

Let the Lesser of them be denoted by x ; then the Greater will be $x + 6$; and so, by the Question, we shall have $xx + 6x = 720$.

But in order to the Resolution of this Equation (in which both the first and second Powers of x are involved)

volved) let Half the Coefficient of x , which (in this Case) is 3, be taken and squared, and let that Square be added to both Sides of the Equation : By which Means it becomes $xx + 6x + 9 = 729$.

Whereof the former Part being now a compleat Square, its Root may, therefore, be extracted ; and will be expressed by the said Half Coefficient joined to x with its proper Sign ; that is, by $x + 3$ (as may be very easily proved by the Multiplication of $x + 3$ into itself ; whence $xx + 6x + 9$, the very Quantity above, is produced). Hence it is evident, that $x + 3 = \sqrt{729} = 27$ (for equal Quantities have equal square Roots) ; and consequently $x = 24$.

QUESTION LXVI.

The Sum of two Numbers being given = 60, and the Sum of their Squares = 1872 ; to find the Numbers.

Let x be the Greater of them ;
then $60 - x$ will be the Lesser :

And therefore $x^2 + 60 - x^2 = 1872$;

Or $x^2 + 3600 - 120x + x^2 = 1872$:

Whence $2x^2 - 120x = -1728$,

and $x^2 - 60x = -864$:

From which, by completing the Square (*as in the last Problem*) we get $x^2 - 60x + 900 (= -864 + 900) = 36$:

And consequently, by taking the Root, $x - 30 = \sqrt{36} = 6$.

Therefore $x = 6 + 30 = 36$; and $60 - x = 24$: Which are the two Numbers that were to be found.

But, to solve the Problem in a more general manner (by Letters) put the Sum of the two Numbers $= a$, the Sum of their Squares $= b$, and the greater Number $= x$, *as before*.

Then will $x^2 + a^2 - 2ax + x^2 = b$, *by the Question*.

Hence $2x^2 - 2ax = b - a^2$,

and $x^2 - ax = \frac{b}{2} - \frac{a^2}{2}$.

Da

Where,

Where, Half the Coefficient of the second Term being $\frac{a}{2}$, the compleated Square will therefore be

$$x^2 - ax + \frac{a^2}{4} \left(= \frac{b}{2} - \frac{a^2}{2} + \frac{a^2}{4} \right) = \frac{b}{2} - \frac{a^2}{4} :$$

From which, by extracting the Root,

$$\text{we have } x - \frac{a}{2} = \sqrt{\frac{b}{2} - \frac{aa}{4}} ;$$

And therefore $x = \sqrt{\frac{b}{2} - \frac{aa}{4}} + \frac{a}{2}$: Which, if a be taken = 60, and $b = 1872$, will come out = 36, the very same as before.

QUESTION LXVII.

To divide the Number 60 (a) into two such Parts, that their Product may be 864 (b).

If x be put for the greater Part, the Lesser will be denoted by $a-x$; and we shall therefore have $ax - xx = b$; by the Conditions of the Question.

This Equation, by changing the Signs of all its Terms (in order to have the highest Power of x affirmative) becomes $xx - ax = -b$.

Whence, by compleating the Square,

$$\text{we have } xx - ax + \frac{aa}{4} = -b + \frac{aa}{4} ;$$

$$\text{and consequently } x - \frac{a}{2} = \sqrt{\frac{aa}{4} - b}.$$

Therefore $x = \sqrt{\frac{aa}{4} - b} + \frac{a}{2} = 36$, the greater Part ;
and $a - x = 24$, the Lesser.

QUESTION

QUESTION LXVIII.

To divide the Number 60 (a) into two Parts, so that the Square of the Greater multiply'd by the Lesser, added to the Square of the Lesser multiply'd by the Greater, may amount to 51840 (b).

If x be the greater Part; then $a-x$ will be the Lesser;
and $x^2 \times a-x + a-x)^2 \times x = b$,

that is, $ax^2 - x^3 + a^2x - 2ax^2 + x^3 = b$,

or, $a^2x - ax^2 = b$:

Whence $x^2 - ax = -\frac{b}{a}$;

and, consequently, $x = \sqrt{\frac{a^2}{4} - \frac{b}{a}} + \frac{a}{2} = 36$.

QUESTION LXIX.

The Sum of two Numbers being given = 20 (a), and the Sum of their Cubes = 2240 (b); to determine the Numbers.

Let x be the Greater; then $a-x$ will be the Lesser;

and therefore $x^3 + a-x)^3 = b$,

that is, $x^3 + a^3 - 3a^2x + 3ax^2 - x^3 = b$:

Whence $3ax^2 - 3a^2x = b - a^3$,

and $x^2 - ax = \frac{b}{3a} - \frac{a^2}{3}$.

Therefore, by compleating the Square,

$x^2 - ax + \frac{aa}{4} \left(= \frac{b}{3a} - \frac{a^2}{3} + \frac{a^2}{4} \right) = \frac{b}{3a} - \frac{a^2}{12}$:

And, by extracting the Root, $x - \frac{a}{2} = \sqrt{\frac{b}{3a} - \frac{aa}{12}}$;

Consequently $x = \frac{a}{2} + \sqrt{\frac{b}{3a} - \frac{aa}{12}} = \frac{20}{2} +$

$\sqrt{37\frac{1}{3} - 33\frac{1}{3}} = 12$.

QUESTION LXX.

To divide the Number 240 (a) into two such Parts, that the greater Part divided by the Lesser, may be to the lesser Part divided by the Greater, in the Proportion of 147 to 75 (or of m to n).

If the greater Part be denoted by x , the Lesser will be expressed by $a-x$; and we shall have

$$\frac{x}{a-x} : \frac{a-x}{x} :: m : n.$$

$$\text{Hence } \frac{nx}{a-x} = \frac{m \times a-x}{x}$$

$$\text{and } \frac{nx^2}{m} = (a-x)^2:$$

From which, by extracting the square Root, on both

$$\text{Sides, } x \sqrt{\frac{n}{m}} = a-x.$$

$$\text{Whence, putting } \sqrt{\frac{n}{m}} = \frac{b}{c},$$

$$\text{we have } bx = ca - cx; \text{ and consequently } x = \frac{ca}{b+c}.$$

$$\text{But, in the Case above proposed, } \sqrt{\frac{n}{m}} \text{ being } =$$

$$\sqrt{\frac{75}{147}} = \sqrt{\frac{25}{49}} = \frac{5}{7}, \text{ we have } b=5, c=7; \text{ and}$$

$$\text{therefore } x = \frac{7 \times 240}{12} = 7 \times 20 = 140.$$

QUESTION

QUESTION LXXI.

Two Workmen A and B were employ'd, by the Day, at different Rates; A, at the End of a certain Number of Days, had 96 Shillings to receive; but B, who play'd 6 of those Days, received, only, 54 Shillings: But, had B worked the whole Time, and A play'd 6 Days, They would have received exactly alike. 'Tis propos'd to find the Number of Days They were employ'd; and what Each had a Day.

Let x be the Number of Days that A work'd; then $x-6$ will be the Days that B work'd.

Moreover $\frac{96}{x}$, will be the Wages of A per Day;

and $\frac{54}{x-6}$, the Wages of B per Day.

Therefore $\frac{54}{x-6} \times x$, is what B would have earn'd, had he work'd the whole Time:

And $\frac{96}{x} \times x-6$, what A would have earn'd had he play'd 6 Days. Which two Values being equal, by the Question,

we have $\frac{54x}{x-6} = \frac{96 \times x-6}{x}$.

Whence, by Reduction, $54x^2 = 96 \times x-6]^2$,

Or, $\frac{9x^2}{16} = x-6]^2$: Therefore, by extracting the square

Root on both Sides, $\frac{3x}{4} = x-6$; and consequently $x=24$.

From which it is evident, that A had 4 Shillings, and B 3 Shillings, a Day.

QUESTION LXXII.

From two Places, at the Distance of 320 (a) Miles, two Persons, A and B, set out, at the same Time, in order to meet each other; A travelled 8 (b) Miles a Day more than B; and the Number of Days in which They met was equal to Half the Number of Miles B went in a Day: 'Tis required to find how far Each travelled to meet the Other.

Let x be the Number of Days in which They met; then, by the { $2x$
Question, { $2x+b$ } will be the Number of Miles { B
went a Day. { A }

Therefore, by multiplying each of These by (x) the Number of Days, we have $2xx$, and $2xx+bx$, for the whole Number of Miles travelled by B and A, respectively:

And consequently $4xx+bx=a$.

$$\text{Hence } xx + \frac{bx}{4} + \frac{bb}{64} = \frac{a}{4} + \frac{bb}{64},$$

$$\text{and } x = \sqrt{\frac{a}{4} + \frac{bb}{64}} - \frac{b}{8} = 8.$$

Therefore $2xx = 128 =$ the Miles travelled by B; and $2xx+bx = 192 =$ Those travelled by A.

QUESTION LXXIII.

Two Messengers, A and B, were dispatched at the same Time, to a Place, at the Distance of 90 (a) Miles; the Former of whom, by riding one Mile an Hour more than the Other, arrived at the End of his Journey one Hour before him: The Question is, to find at what Rate Each travelled per Hour.

If x be the Miles that A rode per Hour; then $x-1$ will be the Miles which B rode per Hour:

Moreover $\frac{a}{x}$ will be the Number of Hours in which A performed

performed the whole Journey; and $\frac{a}{x-1}$ will be the Number of Hours wherein B performed it.

And therefore $\frac{a}{x} = \frac{a}{x-1} - 1$, by the Question,

whence, by Reduction $ax - a = ax - x^2 + x$,

Or $x^2 - x = a$;

Therefore $x^2 - x + \frac{1}{4} = a + \frac{1}{4}$ (by compleating the Square)

and consequently $x = \sqrt{a + \frac{1}{4}} + \frac{1}{2} = 9\frac{1}{2} + \frac{1}{2} = 10$.

QUESTION LXXIV.

To find two Numbers, so that their Sum multiply'd by the Greater may produce 100 times the Lesser, and being multiply'd by the Lesser may produce 64 times the Greater.

Let x denote the greater Number, and y the Lesser.

Then $x + y \times x = 100y$,

And $x + y \times y = 64x$.

Now, the first of these Equations being multiply'd by y , and the Second by x , they become both alike;

and so we have $100y^2 = 64x^2$;

Therefore, by taking the square Root, $10y = 8x$,

and consequently $y = \frac{4x}{5}$: Which Value substituted in

the second Equation, gives $x + \frac{4x}{5} \times \frac{4x}{5} = 64x$;

Or, $\frac{9x}{5} \times \frac{4}{5} = 64$.

Whence $x = \frac{25 \times 16}{9} = 44\frac{4}{9}$; and $y = 35\frac{5}{9}$.

QUESTION

QUESTION LXXV.

To find three Numbers, so that their continual Product divided, by the Sum of each two of them, may quote given Numbers; or (which is the same Thing) to determine the Values of x , y and z , in the underwritten Equations.

$$\frac{xyz}{x+y}=200, \frac{xyz}{x+z}=150, \frac{xyz}{y+z}=120.$$

First, by Multiplication,

$$xyz = 200x + 200y = 150x + 150z = 120y + 120z:$$

Therefore, by Reduction,

$$\left. \begin{array}{l} 50x + 200y = 150z \\ 200x + 80y = 120z \end{array} \right\} \text{ or } \left\{ \begin{array}{l} x + 4y = 3z, \\ 5x + 2y = 3z. \end{array} \right.$$

Hence $x + 4y = 5x + 2y$, and therefore $y = 2x$:

Which, substituted in $3z = x + 4y$, gives $3z = 9x$; and $z = 3x$.

And, by substituting for both y and z , in the Equation

$$\frac{xyz}{x+y}=200, \text{ we get } \frac{x \times 2x \times 3x}{x+2x}=200;$$

that is, $2x^2 = 200$.

Consequently $x = 10$, $y = 20$, and $z = 30$.

QUESTION LXXVI.

To find the Ratio of two Numbers, whose Rectangle is equal to the Square of their Difference.

Let the lesser Number be to the Greater as 1 is to x ; then, if the said lesser Number be denoted by z , the Greater will be expressed by xz , and we shall have $xz \times z = \overline{xz - z}^2$, or $xz^2 = x^2 z^2 - 2xz^2 + z^2$ (by the Question). Whence, dividing the whole by z^2 , there results $x = x^2 - 2x + 1$:

Therefore $x^2 - 3x = -1$;

and consequently $x = \frac{3 + \sqrt{5}}{2} = 2.618 \text{ \&c.}$

Hence

Hence the Ratio of any two Numbers, whose Rectangle is equal to the Square of their Difference, must be that of 2. 618 &c. to Unity.

QUESTION LXXVII.

To find two Numbers, whose Product is 300 (a); so that, if 10 (b) be added to the Lesser, and 8 (c) subtracted from the Greater, the Product of the Sum and Remainder shall, also, be equal to 300 (a).

Let the greater Number be denoted by x , and the Lesser by y ;

then will $\left\{ \begin{array}{l} xy = a \\ x - c \times y + b = a \end{array} \right\}$ by the Problem.

By the last of which $xy + bx - cy - cb = a$.

From whence, the first Equation being subtracted, there rests $bx - cy - cb = 0$:

Therefore $bx = cb + cy$, and $x = \frac{cb + cy}{b}$: Which sub-

stituted in the first Equation, gives $\frac{cby + cy^2}{b} = a$.

From which we have $y^2 + by = \frac{ab}{c}$:

And, by completing the Square, $y^2 + by + \frac{bb}{4} = \frac{ab}{c} + \frac{bb}{4}$.

Whence $y = \sqrt{\frac{ab}{c} + \frac{bb}{4}} - \frac{b}{2} = 15$;

and $x \left(= \frac{a}{y} \right) = 20$.

QUESTION LXXVIII.

To divide 100 into two such Parts, that their Difference may be to their Sum, as their Rectangle to the Difference of their Squares.

Let a represent half the given Number, and x half the Difference of its two Parts; then, the Greater of them

them being expressed by $a+x$, and the Lesser by $a-x$,
it will be $2x : 2a :: \overline{a+x} \times \overline{a-x} : \overline{a+x}^2 - \overline{a-x}^2$;
that is, by Reduction, $x : a :: aa - xx : 4ax$;

Hence $4ax^2 = a \times aa - xx$,

Or $4x^2 = a^2 - x^2$,

and therefore $x = \sqrt{\frac{aa}{5}} = 22,36$.

So that, the greater Part is 72,36; and the Lesser 27,64.

QUESTION LXXIX.

To divide the Number 60 (a) into two such Parts, that
their Product may be to the Sum of their Squares, in
the Ratio of 2 (m) to 5 (n).

Let x be the greater Part :

Then, the lesser Part will be $a-x$,

the Product $ax - xx$,

and the Sum of the Squares $a^2 - 2ax + 2x^2$.

Therefore $m : n :: ax - xx : a^2 - 2ax + 2x^2$;

and so, $2mx^2 - 2max + ma^2 = nax - nx^2$.

Whence, $2mx^2 + nx^2 - 2max - nax = -ma^2$;

that is, $2m+n \times x^2 - 2m+n \times ax = -ma^2$;

Or, $x^2 - ax = -\frac{maa}{2m+n}$.

Therefore $x^2 - ax + \frac{aa}{4} = \left(\frac{aa}{4} - \frac{ma^2}{2m+n} \right) = \frac{aa}{4} \times \frac{n-2m}{n+2m}$,

and $x = \frac{a}{2} \sqrt{\frac{n-2m}{n+2m}} + \frac{a}{2} = 40$.

QUESTION LXXX.

To find two Numbers whose Product shall be 320 (a), and
the Difference of their Cubes to the Cube of their Dif-
ference, as 61 (n) is to Unity.

Let x be the greater Number, and y the Lesser ;
then will $xy = a$,

and $x^3 - y^3 : x - y :: n : 1$, by the Question :

Which

Which, by actually involving $x-y$,

becomes $x^3-y^3 : x^3-3x^2y+3xy^2-y^3 :: n : 1$.

From whence (by subtracting the Consequents from their Antecedents)

we get $3x^2y-3xy^2 : x^3-3x^2y+3xy^2-y^3 :: n-1 : 1$;

Or, which is the same, $3xy \times \overline{x-y} : \overline{x-y}^3 :: n-1 : 1$.

Whence, dividing by $x-y$,

we have $3xy : \overline{x-y}^2 :: n-1 : 1$;

Or, $3a : \overline{x-y}^2 :: n-1 : 1$ (because $xy=a$).

From whence $\overline{x-y}^2 = \frac{3a}{n-1}$;

and consequently $x-y = \sqrt{\frac{3a}{n-1}}$; which put $= b$:

Then from the Equations $xy=a$, and $x-y=b$, the

Value of x will be found $= \frac{\sqrt{b^2+4a}+b}{2} = 20$; and

That of $y = \frac{\sqrt{b^2+4a}-b}{2} = 16$, *Vid. Prob. 65*.

The same otherwise.

Let z denote the Half Sum, and x the Half Difference, of the two Numbers; then the Greater will be expressed by $z+x$, and the Lesser by $z-x$; and we shall therefore

have $\left\{ \begin{array}{l} \overline{z+x} \times \overline{z-x} = a \\ \overline{z+x}^3 - \overline{z-x}^3 = n \times \overline{2x}^3 \end{array} \right\}$ by the Question.

that is $\left\{ \begin{array}{l} z^2 - x^2 = a, \\ 6z^2x + 2x^3 = 8nx^3. \end{array} \right.$

The last of which, divided by $2x$,

gives $3z^2 + x^2 = 4nx^2$.

From whence, the Treble of the first being deducted,

we have $4x^2 = 4nx^2 - 3a$; and consequently $x =$

$\sqrt{\frac{3a}{4n-4}} = 2$.

Hence $z (= \sqrt{a+xx}) = 18$; therefore $z+x = 20$,
and

and $z - x = 16$; which are the two Numbers that were to be found.

QUESTION LXXXI.

A Farmer received 7l. 4s. for a certain Quantity of Wheat, and an equal Sum, at a Price less by 1s. 6d. a Bushel, for a Quantity of Barley, which exceeded That of the Wheat by 16 Bushels: How many Bushels were there of Each?

Put a = the total Value of each Sort of Grain,
 b = the Difference of the Quantities,
 c = the Difference of the Prices *per* Bushel,
 and x = the Number of Bushels of the Wheat:

Then, dividing the whole Price by the Number of Bushels, we have $\frac{a}{x}$ for the Price of the Wheat *per* Bushel: And, in the same manner, the Price of the Barley *per* Bushel will appear to be $\frac{a}{x+b}$.

Therefore $\frac{a}{x} - \frac{a}{x+b} = c$, by the Question.

Whence (by Reduction) $ax + ab - ax = cx^2 + bcx$;

Or, $\frac{ab}{c} = x^2 + bx$.

Consequently $x = \sqrt{\frac{ab}{c} + \frac{bb}{4}} - \frac{b}{2} = 32$, the Number of Bushels of Wheat; and $x + 16 = 48$, the Number of Bushels of Barley.

QUESTION

QUESTION LXXXII.

One bought two Pieces of Cloth of different Sorts, whereof the Finer cost 4 Shillings a Yard more than the Other; so that, for the Finest, he paid 360 Shillings; whereas the Coarsest, which exceeded the Finest by 10 Yards, cost him, only, 320 Shillings: How many Yards were there of each Piece?

Let x be the Number of Yards of the Finest, and y , the Number of Shillings a Yard: Then $x+10$, will be the Length of the coarsest Piece, and $y-4$, its Price *per* Yard.

Hence $\left\{ \begin{array}{l} \frac{xy}{x+10} = \frac{360}{y-4} \end{array} \right\}$ by the Question.

By the last Equation $xy + 10y - 4x = 360$; from whence deducting $xy = 360$, we have $10y - 4x = 0$.

Therefore $10y = 4x$; and $y = \frac{2x}{5}$. Which, being substituted in the first Equation, we get $\frac{2x^2}{5} = 360$: Whence x comes out $= \sqrt{900} = 30$.

QUESTION LXXXIII.

There are two Numbers, whose Rectangle is equal to the Difference of their Squares; and the Sum of their Squares is also equal to the Difference of their Cubes: What are those Numbers?

Let x denote the lesser Number, and let the Greater be in Proportion thereto as y is to Unity; Or, which is the same Thing, let the greater Number be denoted by xy .

Then $\left\{ \begin{array}{l} x \times xy = x^2 y^2 - x^2 \\ x^2 y^2 + x^2 = x^3 y^3 - x^3 \end{array} \right\}$ by the Problem.

Therefore $\left\{ \begin{array}{l} y = \frac{y^2 - 1}{y^2 + 1} \\ y^2 + 1 = xy^3 - x \end{array} \right\}$ by Division.

From

From the first of which Equations, $y^2 - y = 1$;

whence $y^2 - y + \frac{1}{4} = \frac{5}{4}$, and consequently $y = \frac{1 + \sqrt{5}}{2}$

But, by the second Equation $x = \frac{yy + 1}{yyy - 1} = \frac{y + 2}{2y}$ (be-

cause $yy = y + 1$) $= \frac{1}{2} + \frac{1}{y} = \frac{1}{2} + \frac{2}{\sqrt{5} + 1} = \frac{1}{2} + \frac{2\sqrt{5} - 2}{4}$ (by multiplying both the Numerator and De-

nominator by $\sqrt{5} - 1$) $= \frac{\sqrt{5}}{2}$. Therefore the two

Quantities sought are $\frac{1}{2}\sqrt{5}$, and $\frac{5 + \sqrt{5}}{4}$.

QUESTION LXXXIV.

A sets out from London for York, at the same time as B sets out from York for London; and the Rate at which They travel is such, that A, 9 Hours after their meeting arrives at York, and B at London, in 16 Hours after. The Question is, to find in what Time each Traveller performs his Journey.

Let x denote the Number of Hours travelled by Each before the Time of their Meeting on the Road.

Then, since A (*by the former Part of the Question*) goes over the very same Ground in 9 (a) Hours, as B travelled in x Hours, we have, therefore, as $a : x :: x :$

$\frac{xx}{a}$, the Time wherein B travels a Distance equal to

That gone over by A, in x Hours: But (*by the latter Part of the Question*) B, in 16 (b) Hours, travels the very same Distance as A in x Hours. Hence it is evi-

dent that $\frac{xx}{a}$ and b are equal to each Other:

And consequently that $x = \sqrt{ab} = 12$.

Therefore

Therefore A performs the Journey in 21 Hours, and B in 28 Hours.

QUESTION LXXXV.

The Sum of 190 l. was divided between three Persons; whose Shares were in Geometrical Proportion, and the greatest of them exceeded the least by 50 l. What were all the several Shares?

If x be put for the Least of them, then the Greatest will be $x+50$; the Sum of which two, subtracted from (190) the Whole, leaves $140-2x$ for the Mean Share: Therefore, $x : 140-2x :: 140-2x : x+50$, by the Question.

And consequently $\overline{140-2x}^2 = x \times \overline{x+50}$; that is, $19600-560x+4x^2 = x^2+50x$.

Whence $3x^2-610x=-19600$,

and $x^2 - \frac{610x}{3} = -\frac{19600}{3}$.

Hence, by completing the Square,

$$x^2 - \frac{610x}{3} + \frac{93025}{9} \left(= -\frac{19600}{3} + \frac{93025}{9} \right) = \frac{34225}{9};$$

And, by extracting the Root, $x - \frac{305}{3} = \pm \frac{185}{3}$.

Therefore $x = \frac{305-185}{3} = 40$: And so the other two Shares are 60 and 90 Pounds.

QUESTION LXXXVI.

Two Notes, One of 120 l. payable in 6 Months, and the Other of 150 l. payable in 9 Months, were discounted for 8 l. 10 s. What Rate of Interest were They discounted at?

Let x denote the Interest of One Pound for 12 Months:

Then the Amount of 1 l. in 6 Months being $1 + \frac{x}{2}$, and,

E

in

in 9 Months $1 + \frac{3x}{4}$ the *Present Value* of the Bill, due

at the End of 6 Months, will therefore be $\frac{120}{1 + \frac{1}{2}x}$; and

That of the Bill, due at the End of 9 Months, $\frac{150}{1 + \frac{3}{4}x}$.

Whence, we have $\frac{120}{1 + \frac{1}{2}x} + \frac{150}{1 + \frac{3}{4}x} (= 120 + 150 - 8.5) = 261,5$ (by the *Question*)

Which, by Reduction, becomes $120 + 90x + 150 + 75x = 261,5 \times 1 + \frac{5}{4}x + \frac{3}{8}x^2$;

$$\text{or, } 270 + 165x = \frac{261,5 \times 8 + 10x + 3x^2}{8}.$$

$$\text{Therefore } \frac{270 + 165x \times 8}{261,5} = 3x^2 + 10x + 8,$$

$$\text{that is, } \frac{2160}{261,5} + \frac{1320x}{261,5} = 3x^2 + 10x + 8:$$

From which we have

$$x^2 + \frac{2590}{3 \times 523} \times x = \frac{136}{3 \times 523}.$$

Whence, by completing the Square, &c.

$$x \text{ is found} = \sqrt{\frac{136}{1569} + \frac{1295}{1569}}^2 - \frac{1295}{1569} = \frac{\sqrt{136 \times 1569 + 1295 \times 1295} - 1295}{1569} = 0,05093. \text{ Which}$$

multiply'd by 100, gives 5,093 *l.* or *l.* 5 : 1 : 10 $\frac{1}{2}$, nearly, for the Rate *per Cent*, at which the Notes were discounted.

QUESTION

QUESTION LXXXVII.

A and B take, in Trade, 5940 per Annum, each; but A, whose Profits are 2 per Cent. greater than Those of B, clears 100*l.* per Annum more than B: What are the Profits of Each, per Cent? And What do They clear per Annum?

Let $c = 100$ (Pounds),

$b (= 5940) =$ the whole Sum taken by Each,

$d (= 100*l.*) =$ the given Difference of their Gains,

$x =$ What A gains per Cent.

$x - a (= x - 2) =$ What B gains per Cent.

Now, since A, in taking $c + x$ Pounds, gains x Pounds,

it will be $c + x : x :: b : \frac{bx}{c + x}$, the whole Gain of A:

And, in the same manner, we have,

$c + x - a : x - a :: b : \frac{b \times x - a}{c + x - a}$, the whole Gain of B:

Therefore $\frac{bx}{c + x} - \frac{b \times x - a}{c + x - a} = d$, by the Question.

Hence $abc = x^2 + 2cx - ax - ac + cc \times d$;

and consequently $x^2 + 2c - a \times x = \frac{abc}{d} + ac - cc$.

Whence, by completing the Square,

$$x^2 + 2c - a \times x + \left| \frac{2c - a}{2} \right|^2 = \frac{abc}{d} + \frac{aa}{4};$$

$$\text{and therefore } x = \sqrt{\frac{abc}{d} + \frac{aa}{4} - \frac{2c - a}{2}} = 10.$$

From which it appears that A gained 10 per Cent. and cleared 540 *l.* per Annum; and that B gained 8 per Cent. and cleared 440 *l.* per Annum.

QUESTION LXXXVIII.

Of four Numbers in Geometrical Progression, there is given the Sum of the two Least $= 20$ (a) and the Sum of the two Greatest $= 45$ (b); to find the Numbers.

Let x denote the first Number, and y the Third; then the Second being expressed by $a-x$, and the Fourth by $b-y$, the four Terms of the Progression, placed in Order, will stand thus, x , $a-x$, y , $b-y$, Whence, by the Nature of Proportionals,

$$\text{we have } \begin{cases} x \times b-y = a-x \times y, \\ xy = a-x^2. \end{cases}$$

From the first of which Equations $y = \frac{bx}{a}$: And, by

substituting in the Second, we have $\frac{bx^2}{a} = a-x^2$:

Whence, by extracting the Square Root on both Sides,

$$x \sqrt{\frac{b}{a}} = a-x;$$

$$\text{and consequently } x = \frac{a}{1 + \sqrt{\frac{b}{a}}} = 8.$$

Therefore 8, 12, 18, and 27, are the four Numbers that were to be found.

QUESTION LXXXIX.

The Sum of 700l. (a) was divided among four Persons A, B, C and D; whose Shares were in Geometrical Progression; and the Difference between the Greatest and Least, was to the Difference between the two Means, as 37 (m) to 12 (n): What were all the several Shares?

Let the Share of A, or the first Term of the Progression, be denoted by x , and let the Common Ratio be That of 1 to y :

Then

Then $x + xy + xy^2 + xy^3 = a$ } *by the Question.*
 and $xy^3 - x : xy^2 - xy :: m : n$

From which Proportion, we have

$$y^3 - 1 = \overline{y-1} \times \frac{my}{n}, \text{ or } y^2 + y + 1 = \frac{my}{n} \text{ (by dividing the whole by } y-1 \text{).}$$

$$\text{Hence } y \text{ is found } = \frac{m-n + \sqrt{mm - 2mn - 3nn}}{2n} = \frac{25+7}{24}$$

$$= \frac{4}{3}: \text{ Whence } x \left(= \frac{a}{1+y+y^2+y^3} \right) \text{ is given } =$$

$$\frac{27 \times 700}{27 + 36 + 48 + 64} = 108. \text{ Therefore the four Shares are } 108, 144, 192, \text{ and } 256 l.$$

QUESTION XC.

Of four Numbers in Arithmetical Progression, the Sum of the Squares of the two Means is given = 400 (a); and the Sum of the Squares of the two Extremes = 464 (b): To determine the Numbers.

If x be put for the lesser Extreme, and y for the Common Difference; then the four Numbers will be expressed by,

$x, x+y, x+2y$, and $x+3y$ respectively; and we

shall therefore have $\left\{ \begin{array}{l} \overline{x+y}^2 + \overline{x+2y}^2 = a \\ x^2 + \overline{x+3y}^2 = b \end{array} \right\}$ or

$$\left\{ \begin{array}{l} 2x^2 + 6xy + 5y^2 = a \\ 2x^2 + 6xy + 9y^2 = b \end{array} \right\} \text{ by the Question.}$$

Whence, by Subtraction, $4y^2 = b-a$:

$$\text{And therefore } y = \frac{\sqrt{b-a}}{2} = 4.$$

$$\text{But, by the first Equation, } x^2 + 3xy = \frac{a}{2} - \frac{5yy}{2}:$$

From which (looking upon y as known) we get

$$x = \sqrt{\frac{a}{2} - \frac{yy}{4} - \frac{3y}{2}} = 8.$$

Therefore 8, 12, 16, and 20, are the four Numbers that were to be found.

QUESTION XCI.

The Sum of four Numbers, in Arithmetical Progression, being given = 56 (b) and the Sum of their Squares = 864 (c); to find the Numbers.

If Half the Sum of the two middle Numbers be denoted by a , and Half their Difference by x , the Numbers themselves will be expressed by $a-x$, and $a+x$: And we shall have

$$\left\{ \begin{array}{l} \overline{a-3x} + \overline{a-x} + \overline{a+x} + \overline{a+3x} = b \\ \overline{a-3x}^2 + \overline{a-x}^2 + \overline{a+x}^2 + \overline{a+3x}^2 = c \end{array} \right\} \begin{array}{l} \text{by the Nature of} \\ \text{the Question.} \end{array}$$

Hence, by Reduction, $4a = b$, and $4aa + 20xx = c$.

$$\text{Therefore } a = \frac{b}{4} = 14; \text{ and } x = \sqrt{\frac{c}{20} - \frac{aa}{5}} = 2.$$

From which the Numbers themselves are given; being 8, 12, 16 and 20.

By the same way of proceeding the Problem may be resolved, when the Progression is supposed to consist of 6, 8, 10, or any other, even, Number (n) of Terms.

For the Sum of the Squares of the two Means ($a-x$, and $a+x$) being $= 2 \times \overline{aa + xx}$; and the Sum of the Squares of the two Terms ($a-3x$ and $a+3x$) next adjacent to them $= 2 \times \overline{aa + 9xx}$; also the Sum of the Squares of the two Terms ($a-5x$ and $a+5x$) next adjacent to these last being $= 2 \times \overline{aa + 25xx}$, &c. &c. it follows that

$$2 \times \overline{aa + xx} + 2 \times \overline{aa + 9xx} + 2 \times \overline{aa + 25xx} + \text{\&c.} = c.$$

Which Equation, by putting $1 + 9 + 25 + 36 + \text{\&c.}$

(continued to $\frac{n}{2}$ Terms) $= f$, is reduced to $naa + 2fxx$

$= c.$

Whence

Whence $x = \sqrt{\frac{c - naa}{2f}}$; from which, as a is given $= \frac{b}{n}$, the Value of x will also be known.

QUESTION XCII.

Having the Sum (b) and the Sum of the Squares (c) of five Numbers, in Arithmetical Progression; to determine the Progression.

Let a denote the middle Number, and x the Common Difference; then the five Terms of the Progression will be, $a - 2x$, $a - x$, a , $a + x$, and $a + 2x$; whence we have $5a = b$, and $5aa + 10xx = c$, by the Conditions of the Question. From which a is found $= \frac{b}{5}$, and $x (=$

$$\sqrt{\frac{c}{10} - \frac{aa}{2}} = \sqrt{\frac{c}{10} - \frac{bb}{50}}.$$

After the same Manner the Problem may be investigated, when any odd Number (n) of Terms is propounded: For the Sum of the Squares of the 2 Terms ($a - x$ and $a + x$) adjacent to the middle One being $= 2 \times aa + xx$; and That of the two Terms ($a - 2x$ and $a + 2x$) next to These $= 2 \times aa + 4xx$, &c. &c. it is evident therefore that

$$aa + 2 \times aa + xx + 2 \times aa + 4xx + 2 \times aa + 9xx + \&c. = c.$$

Which Equation, by putting $1 + 4 + 9 + 16 + \&c.$ (to $\frac{n-1}{2}$ Terms) $= f$, becomes $naa + 2fxx = c.$

Hence $x = \sqrt{\frac{c - naa}{2f}}$; from which, as a is given $= \frac{b}{n}$, all the Terms of the Progression will be known.

By a like Process, if, instead of the Sum of the Squares, the Sum of the Cubes, or Biquadrates, be given, the Problem may be resolved.

QUESTION XCIII.

A Traveller, bound to a certain Place at the Distance of 140 Miles, goes 26 Miles the first Day, 24 Miles the second Day, 22 Miles the Third; and so on, in Arithmetical Progression, decreasing 2 Miles every Day. In how many Days will he arrive at the End of his Journey?

Put $b = 140$, the whole, given, Distance,
 $c = 26$, the Part thereof travelled in the first Day,
 $d = 2$, the Common Difference by which each Day's Journey decreases,
 and $x =$ the Number of Days wherein the *whole* Journey is perform'd.

Then, from the Nature of the Question, it is evident, that, the last Day's Journey will fall short of the First, by $x-1$ times the Common Difference (d); and is, therefore, truly expressed by $c - \overline{x-1} \times d$:

But it is well known that the Sum of any Arithmetical Progression is equal to the Sum of the first and last Terms multiply'd by Half the Number of the Terms:

Hence we have $\overline{c + c - \overline{x-1} \times d} \times \frac{x}{2}$ for the whole Distance travelled; and consequently this Equation,

$$\overline{2c - \overline{x-1} \times d} \times \frac{x}{2} = b.$$

Whence $2cx - dx^2 + dx = 2b$,

$$\text{and } xx - \frac{2c+d}{d} \times x = -\frac{2b}{d} :$$

$$\text{From which } x = \frac{+}{-} \sqrt{\frac{2c+d}{4dd}} - \frac{2b}{d} + \frac{2c+d}{2d} = 7.$$

QUESTION

QUESTION XCIV.

After A, who travelled at the Rate of 4 Miles an Hour, had been set out two Hours and three Quarters, B set out to overtake him; and in order thereto went $4\frac{1}{2}$ Miles the first Hour, $4\frac{3}{4}$ the Second, 5 the Third; and so on, gaining a Quarter of a Mile every Hour. How many Hours did it take him to come up with A?

Put $a = 4$, the Distance travelled by A, per Hour.

$b = 11$, the Distance gone by A before B set out,

$c = 4\frac{1}{2}$, the Miles travelled by B in the first Hour,

$d = \frac{1}{4}$ of a Mile, the Common Excess,

and $x =$ the Number of Hours required.

Then it is evident (from the preceding Problem) that the Distance travelled by B, in the last Hour, will be $c +$

$\overline{x-1} \times d$; and that $\overline{2c + x-1 \times d} \times \frac{x}{2}$ will express the

whole Distance travelled by B in x Hours.

But the Distance of A in x Hours being ax , the whole Distance travelled by A will therefore be $ax + b$; which being equal to That of B (by the Question) we thence have,

$$\overline{2c + x-1 \times d} \times \frac{x}{2} = ax + b;$$

and consequently $2cx + dx^2 - dx = 2ax + 2b$;

$$\text{or } xx + \frac{2c-2a-d}{d} \times x = \frac{2b}{d}.$$

Which (by making $\frac{2c-2a-d}{d} = f$) gives

$$x = \sqrt{\frac{2b}{d} + \frac{ff}{4}} - \frac{f}{2} = 8, \text{ the Number of Hours required,}$$

QUESTION

QUESTION XCV.

To find four Numbers in Arithmetical Progression, which being increased by 2, 4, 8 and 15, respectively, the Sums shall be in Geometrical Progression.

Let x denote the least Number, and y the Common Difference. Then the four Numbers will be expressed by x , $x+y$, $x+2y$, and $x+3y$;

and the four specified Sums,

by $x+2$, $x+y+4$, $x+2y+8$, and $x+3y+15$.

Whence, by the Nature of Geometrical Proportionals,

we have $\left\{ \begin{array}{l} \overline{x+2 \times x+2y+8} = \overline{x+y+4}^2, \\ \overline{x+2 \times x+3y+15} = \overline{x+y+4 \times x+2y+8}, \end{array} \right.$

that is $\left\{ \begin{array}{l} y^2+4y=2x \\ 2y^2+10y+2=5x \end{array} \right. \left\{ \begin{array}{l} \text{by Multiplication,} \\ \text{and Transposition.} \end{array} \right.$

Hence $5y^2 + 20y = 4y^2 + 20y + 4$;

therefore $y^2=4$, and $y=2$:

From which $x=6$; and so the other three Numbers are 8, 10, and 12 respectively :

For 6, 8, 10 and 12 are in Arithmetical Progression; and $6+2$, $8+4$, $10+8$, and $12+15$, are in Geometrical Progression.

QUESTION XCVI.

The Rectangle (a) and the Sum of the Cubes (b) of two Numbers being given; to determine the Numbers.

Let x denote the greater, and y the lesser Number :

Then will $\left\{ \begin{array}{l} xy = a \\ x^3 + y^3 = b \end{array} \right\}$ by the Question.

Whence, by Involution, $x^3 y^3 = a^3$,

and $x^6 + 2x^3 y^3 + y^6 = b^2$.

Let the Quadruple of the former of these two Equations be subtracted from the Latter, and you will have $x^6 - 2x^3 y^3 + y^6 = b^2 - 4a^3$;

and,

and, by extracting the square Root, on both Sides,
 $x^3 - y^3 = \sqrt{b^2 - 4a^3}$. Which, added to the first Equation,
 gives $2x^3 = b + \sqrt{b^2 - 4a^3}$.

Consequently $x = \sqrt[3]{\frac{1}{2}b + \frac{1}{2}\sqrt{bb - 4a^3}}$;

and $y (= \frac{a}{x}) = \sqrt[3]{\frac{1}{2}b + \frac{1}{2}\sqrt{bb - 4a^3}}$.

The same otherwise.

Since $xy = a$, we have $y = \frac{a}{x}$; and therefore

$x^3 + \frac{a^3}{x^3} = b$, by the Question.

Whence $x^6 + a^3 = bx^3$,

or, $x^6 - bx^3 = -a^3$.

Therefore, by compleating the Square,

$x^6 - bx^3 + \frac{bb}{4} (= \frac{bb}{4} - a^3) = \frac{bb - 4a^3}{4}$;

and, by extracting the Root, $x^3 - \frac{b}{2} = \frac{\sqrt{bb - 4a^3}}{2}$.

Hence $x^3 = \frac{1}{2}b + \frac{1}{2}\sqrt{bb - 4a^3}$;

and consequently the Values of x and y , *the very same as before.*

From either of the above Solutions, a General Theorem, for the Resolution of Cubic Equations (according to the Manner of *Cardan*) is very easily deducible.

For, by putting $z = x + y$, and involving each Side to the third Power, we have $z^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 + 3xyx + y = x^3 + y^3 + 3axz$ (by substituting for xy and $x + y$, their Equals, a and z). From whence, by Transposition,

$z^3 - 3az (= x^3 + y^3) = b$.

But it appears, from above, that $z (= x + y)$

is

$$is = \sqrt[3]{\frac{1}{2}b + \frac{1}{2}\sqrt{bb-4a^3}} + \sqrt[3]{\frac{1}{2}b + \frac{1}{2}\sqrt{bb-4a^3}}^a;$$

Which Value, therefore, is the true Root of the Equation $z^3 - 3az = b$.

From which the Root of the Equation $z^3 + cz = b$ (where the second Term is positive) will also be given, by assuming $-3a=c$, and substituting for a : Whence z is found

$$= \sqrt[3]{\frac{b}{2} + \sqrt{\frac{bb}{4} + \frac{c}{3}}} - \sqrt[3]{\frac{b}{2} + \sqrt{\frac{bb}{4} + \frac{c}{3}}}^{\frac{1}{3}c}.$$

QUESTION XCVII.

The Difference of two Numbers being given = 4, and the Sum of their Cubes = 2240; to determine the Numbers.

Let x denote their Half Sum, and d their Half Difference, then the Greater being $x + d$, and the Lesser $x - d$, we have

$$\overline{x+d}^3 + \overline{x-d}^3 = 2240,$$

that is $2x^3 + 6d^2x = 2240,$

$$\text{or, } x^3 + 3d^2x = \frac{2240}{2}.$$

$$\text{Put } c = 3d^2 (=12) \text{ and } b = \frac{2240}{2} (=1120)$$

then it will become $x^3 + cx = b$:

From whence by the general Theorem, in the last Problem, $x =$

$$\sqrt[3]{\frac{b}{2} + \sqrt{\frac{bb}{4} + \frac{c}{3}}} - \sqrt[3]{\frac{b}{2} + \sqrt{\frac{bb}{4} + \frac{c}{3}}}^{\frac{1}{3}c} = 10.$$

Therefore 12 ($=x + d$) and 8 ($=x - d$) are the two Numbers that were to be found.

QUESTION

QUESTION XCVIII.

'Tis propos'd to divide the Number 24 (2a) into two such Parts, that the Difference of their Cubes may be 3584 (2b).

Let $a+x$ express the greater Part, and $a-x$ the Lesser;
 then will $\overline{a+x}^3 - \overline{a-x}^3 = 2b$;
 that is, $6a^2x + 2x^3 = 2b$;
 or $x^3 + 3a^2x = b$.

Put $c = 3a^2$, and $\sqrt[3]{\frac{b}{2} + \sqrt{\frac{bb}{4} + \frac{c}{3}}} = r$:

Then will $x^3 + cx = b$,

and $x = r - \frac{\frac{1}{3}c}{r} = 4$, by the Theorem in the preceding Page. Whence 8 and 16 are the two required Numbers.

QUESTION XCIX.

The Sum of the Squares of two Numbers being given = 208 (g) and the Sum of their Cubes = 2240 (h); to find the Numbers.

Let the greater Number be denoted by $x+y$, and the Lesser by $x-y$:

Then will $\left\{ \begin{array}{l} 2x^2 + 2y^2 = g \\ 2x^3 + 6xy^2 = h \end{array} \right\}$ by the Question.

From the first of these Equations, multiply'd by $3x$, let the Second be subtracted ;

whence you will have $4x^3 = 3gx - h$;

and consequently $x^3 - \frac{3gx}{4} + \frac{h}{4} = 0$;

that is, in Numbers, $x^3 - 156x + 560 = 0$;

The Roots of which Equation (by Sect. 12 of my Treatise of Algebra) will be found to be 10, 4, and -14; whereof the First, only, is for our Purpose: By Means of which we get $y (= \sqrt{g - x^2}) = 2$.

Therefore

Therefore 12 and 8 are the two Numbers that fulfil the Conditions of the Problem.

Note. The Resolution of the above Equation, by the Theorem referred to in the preceding Examples, is impossible; because the square Root of a negative Quantity is to be extracted; as, upon Trial, will be found.

QUESTION C.

The Sum (a) and the continual Product (b) of four Numbers, in Geometrical Progression, being given; to determine the Numbers.

If the lesser of the two Means be represented by $x-y$, and the greater by $x+y$, we shall have

$$\left. \begin{aligned} \frac{x-y}{x+y} + x-y + x+y + \frac{x+y}{x-y} &= a \\ \text{and } (x-y)^2 \times (x+y)^2 &= b \end{aligned} \right\} \text{by the Question.}$$

From the second of these Equations, by putting $c = \sqrt{b}$, and extracting the Square Root, there comes out

$$(x-y) \times (x+y) (=xx-yy) = c.$$

Moreover, from the First, we get

$$\frac{x-y}{x+y} + 2x \times \frac{x-y}{x+y} \times \frac{x+y}{x-y} + \frac{x+y}{x-y} = a \times \frac{x-y}{x+y} \times \frac{x+y}{x-y};$$

$$\text{or, } 2cx + \frac{x-y}{x+y} + \frac{x+y}{x-y} = ac \text{ (by Substitution)}$$

$$\text{that is, } 2cx + 2x^3 + 6xy^2 = ac;$$

$$\text{or } 2cx + 2x^3 + 6x \times xx - c = ac \text{ (because } yy = xx - c).$$

$$\text{Hence } x^3 - \frac{cx}{2} = \frac{ac}{8} : \text{ From which, by the Theorem}$$

in Page 60, the Value of x may be found.

QUESTION

QUESTION CI.

*The Compound Interest of a certain Sum of Money, put out for 4 Years, amounted to 344*l.* $\frac{1}{100}$; but the simple Interest thereof, for the same Time, and at the same Rate, would have been only 320*l.* What was the Sum put out? and what the Rate of Interest?*

Put $a=344.81$, and $b=320$; and let x denote the Interest of 1*l.* for one Year.

Therefore, since the Simple Interest of 1*l.* for 4 Years is $4x$, and the Compound Interest $(1+x)^4 - 1$, or $4x + 6x^2 + 4x^3 + x^4$, we have, as $4x + 6x^2 + 4x^3 + x^4 : 4x :: a : b$, by the Nature of the Question:

and consequently $6x + 4x^2 + x^3 = \frac{4a-4b}{b} = 0.310125$.

From the Resolution of which Equation, x will be found $=.05$: Therefore the Rate of Interest was 5 *per Cent.* and the Sum put out 1600*l.*

QUESTION CII.

The Sum (s) and the Product (p) of any two Numbers being given, to find the Sum of the Squares, Cubes, Biquadrates, &c. of those Numbers.

Let the two Numbers be denoted by x and y :

Then $x+y=s$
and $xy=p$ } by the Question.

The former of which Equations, squared, gives $x^2 + 2xy + y^2 = s^2$; from whence, the Double of the Latter being subtracted, we get $x^2 + y^2 = s^2 - 2p$, the sum of the Squares.

Let this Equation be multiply'd by $x+y=s$, and there arises $x^3 + xy^2 + yx^2 + y^3 = s^3 - 2sp$,

or, $x^3 + xy \times x+y + y^3 = s^3 - 2sp$,

that is, $x^3 + p \times s + y^3 = s^3 - 2sp$ (because $xy=p$, and $x+y=s$).

Therefore

Therefore $x^3 + y^3 = s^3 - 3sp$, the Sum of the Cubes. Again, multiply this last Equation by $x+y=s$,

and you will have $x^4 + xy \times x^2 + y^2 + y^4 = s^4 - 3s^2p$,

or $x^4 + p \times s^2 - 2p + y^4 = s^4 - 3s^2p$ (because $x^2 + y^2 = s^2 - 2p$).

Whence $x^4 + y^4 = s^4 - 4s^2p + 2p^2$, the Sum of the fourth Powers.

Multiply, again, by $x+y=s$, and you will have

$$x^5 + xy \times x^3 + y^3 + y^5 = s^5 - 4s^3p + 2sp^2,$$

$$\text{or, } x^5 + p \times s^3 - 3sp + y^5 = s^5 - 4s^3p + 2sp^2,$$

and therefore $x^5 + y^5 = s^5 - 5s^3p + 5sp^2$, the sum of the fifth Powers.

From whence the Law of Continuation is manifest; being such, that the Sum of the next superior Powers will, *always*, be obtained by multiplying the Sum of the Powers last found by s , and subtracting the Sum of the preceding Ones drawn into p , from the Product.

So that the Sum of the n^{th} Powers (expressed in a general Manner)

$$\text{or } x^n + y^n, \text{ will be } = s^n - ns^{n-2}p + n \times \frac{n-3}{2} s^{n-4}p^2 - \\ n \times \frac{n-4}{2} \times \frac{n-5}{3} s^{n-6}p^3 + n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4} s^{n-8}p^4 - \text{Ec.}$$

QUESTION CIII.

The Sum (a) and the Sum of the Squares (b) of four Numbers, in Geometrical Progression being given; to find the Numbers.

If x and y be assumed to represent the two middle Numbers, then, from the Nature of continued Proportionals, the two Extremes will be expressed by $\frac{xx}{y}$ and $\frac{yy}{x}$: And so

$$\text{we shall have } \frac{x^2}{y} + x + y + \frac{y^2}{x} = a,$$

$$\text{and } \frac{x^4}{y^2} + x^2 + y^2 + \frac{y^4}{x^2} = b.$$

Put

Put $x+y=u$, and $xy=z$:

Then, from the first Equation, $\frac{x^2}{y} + \frac{y^2}{x} = a-u$,

and therefore $x^3 + y^3 = xy \times \overline{a-u}$.

But, because the Sum of the two Means (x and y) is expressed by u , and their Rectangle by z , it is evident, from *Question 102*, that the Sum of their Squares ($x^2 + y^2$) will be exhibited by $u^2 - 2z$, and the Sum of their Cubes ($x^3 + y^3$) by $u^3 - 3uz$.

And, in like manner, the Sum of the two Extremes $\left(\frac{x^2}{y} + \frac{y^2}{x}\right)$ being denoted by $a-u$, and their Rectangle

by z , the Sum of their Squares will be $= \overline{a-u}^2 - 2z$, and the Sum of their Cubes $= \overline{a-u}^3 - 3a-3u \times z$ (but this Last is of no Use in the present Case.)

Hence, by substituting these several Values in our second, and last Equations,

we get $\overline{a-u}^2 - 2z + \overline{u^2 - 2z} = b$,

and $u^3 - 3uz = z \times \overline{a-u}$;

that is, by Reduction,

$$2u^2 - 2au + a^2 - b = 4z,$$

and $u^3 = 2u + a \times z$:

And, consequently, $\overline{2u+a} \times \overline{2u^2-2au+a^2-b} = 4u^3$:

Whence, by Reduction, $u^2 + \frac{bu}{a} = \frac{aa-b}{2}$;

and therefore $u = \sqrt{\frac{aa-b}{2} + \frac{bb}{4aa} - \frac{b}{2a}}$.

From which the several Values of z , x , y will also become known.

QUESTION CIV.

The Sum (a) and the Sum of the Cubes (c) of four Numbers in continued Geometrical Proportion being given; to determine the Numbers.

Let the Notation of the preceding Problem be retained; then our two Equations (*in this Case*)

F

will

will be $\frac{x^2}{y} + x + y + \frac{y^2}{x} = a$,

and $\frac{x^6}{y^3} + x^3 + y^3 + \frac{y^6}{x^3} = c$.

But, it appears, *from Thence*, that the Sum of the Cubes ($x^3 + y^3$) of the two Means is $= u^3 - 3uz$; and that the

Sum of the Cubes $\left(\frac{x^2}{y} + \frac{y^2}{x}\right)$ of the two Extremes is $=$

$\overline{a-u}^3 - 3a - 3u \times z$: Therefore our last Equation, by substituting these Values, becomes $u^3 + \overline{a-u}^3 - 3az = c$.

And, it appears, *by the last Problem*, that the first Equation (by a like Substitution)

is reduced to $u^3 = \overline{2u+a} \times z$.

Hence, by exterminating z out of the two Equations

thus derived, and putting, $\frac{a^2}{3} - \frac{c}{3a} = d$, we obtain

$$\overline{2u+a} \times \overline{u^2-au+d} = u^3;$$

$$\text{or } u^3 - au^2 - \overline{aa-2d} \times u + ad = 0.$$

From whence, the Value of u being found, the rest of the Quantities will be known.

In the same Manner the Problem may be resolved, when (instead of the Sum of the Cubes) the Sum of the 4th, 5th, or 6th, &c. Powers is given.

For the Sum of the n^{th} Powers of the two Means (or $x^n + y^n$) being, *universally*, $= u^n - nu^{n-2}z + n \times \frac{n-3}{2}$

$u^{n-2}z^2$ &c. (See Question 102.)

and the Sum of the n^{th} Powers of the two Extremes

$$= \overline{a-u}^n - n \times \overline{a+u}^{n-2} z + n \times \frac{n-3}{2} \overline{a-u}^{n-4} z^2, \text{ \&c. (since}$$

the Sum of the Roots is here $= a-u$);

we therefore have $u^n + \overline{a-u}^n - nz \times u^{n-2} + \overline{a-u}^{n-2} + \frac{n}{1} \times \frac{n-3}{2} z^2 \times u^{n-4} + \overline{a-u}^{n-4} - \text{\&c.} = c$: Which

Equation, by writing $\frac{u^3}{\overline{2u+a}}$ instead of its Equal z , be-

comes

comes $u^n + \overline{a-u}^n - \frac{nu^3}{2u+a} \times \overline{u^{n-2} + a-u}^{n-2} + \frac{n \times n-3}{2} \times \frac{u^6}{\overline{2u+a}^2} \times \overline{u^{n-4} + a-u}^{n-4} + \text{Ec.} = c.$ Whence u may be found.

QUESTION CV.

Having given the Sum (a) and the Sum of the Squares (b) of five Numbers in Geometrical Progression; to determine the Progression.

Let x , z , and y denote the three middle Numbers, taken in Order: Then $\frac{xx}{z}$ will be the first Number, and $\frac{yy}{z}$ the Last; and we shall have

$$\left\{ \begin{array}{l} \frac{xx}{z} + x + z + y + \frac{yy}{z} = a, \\ \frac{x^4}{z^2} + x^2 + z^2 + y^2 + \frac{y^4}{z^2} = b \end{array} \right\} \text{by the Question.}$$

Put $u = x + y$; then, from the first Equation,

$$\frac{xx}{z} + \frac{yy}{z} = a - u - z.$$

Therefore, seeing the Sum of the two Extremes is expressed by $a - u - z$, and their Rectangle by z^2 (from the Nature of Proportionals) the Sum of their Squares will be $= \overline{a-u-z}^2 - 2z^2$ (by Question 102).

Moreover, the Sum $(x+y)$ of the two Terms adjacent to the Middle one being $= u$, and their Rectangle $= z^2$, the Sum of their Squares $(x^2 + y^2)$ will therefore be $= u^2 - 2z^2$ (by the Same).

And so, by substituting these Values above,

$$\text{we get } \overline{a-u-z}^2 - 2z^2 + u^2 - 2z^2 + z^2 = b,$$

$$\text{and } \frac{u^2 - 2z^2}{z} = a - u - z.$$

Whence $a^2 - 2au - 2az + 2u^2 + 2uz - 2z^2 = b$,
and $az - u^2 - uz + z^2 = 0$.

The Double of which last Equation, added to the Former, gives $a^2 - 2au = b$; whence $u = \frac{a}{2} - \frac{b}{2a}$.

From which, and the Equation $az - u^2 - uz + z^2 = 0$, the value of z will also become known.

QUESTION CVI.

The Sum (a) and the Sum of the Cubes (b) of five Numbers, in continued Geometrical Proportion being given; to find the Numbers.

Retaining the Notation of the last Problem, and proceeding in the same Manner, we have

$$\overline{a-u-z}^3 - \overline{3a-3u-3z} \times z^2 + u^3 - 3uz^2 + z^3 = b;$$

and $az - u^2 - uz + z^2 = 0$ (as before).

The first of which Equations, by Reduction, becomes

$$a^3 - 3a^2 \times \overline{u+z} + 3a \times \overline{u^2+2uz-z^2} - 3u^2z - 3uz^2 + 3z^3 = b;$$

From whence the other Equation, multiply'd by $3z$, being subtracted, there remains

$$a^3 - 3a^2 \times \overline{u+z} + 3a \times \overline{u^2+2uz-z^2} = b;$$

$$\text{therefore } u^2 + 2uz - z^2 - a \times \overline{u+z} = \frac{b}{3a} - \frac{a^2}{3}.$$

But, by the second Equation, $u^2 + 2uz - z^2 = az + uz$;

$$\text{whence, by Substitution, } az + uz - a \times \overline{u+z} = \frac{b}{3a} - \frac{a^2}{3};$$

$$\text{that is, } uz - az = \frac{b}{3a} - \frac{a^2}{3};$$

$$\text{or } az - uz = d \left(\text{by putting } \frac{aa}{3} - \frac{b}{3a} = d \right).$$

From which, the second Equation being subtracted, there results

$$u^2 - z^2 = d: \text{ Wherein let } \left(\frac{az-d}{z} \right) \text{ the Value of } u$$

(found from the former Equation) be now substituted,
and

and we shall have $\frac{az-d^2}{z^2} - z^2 = d$; and consequently

$$z^4 - \overline{aa-d} \times z^2 + 2adz - d^2 = 0.$$

Whence z will be found.

QUESTION CVII.

The Sum (a), the Sum of the Squares (b), and the Sum of the Cubes (c), of any three Numbers being given; to determine the Numbers.

Let them be denoted by x , y and z ; then

since $\left\{ \begin{array}{l} x + y + z = a \\ x^2 + y^2 + z^2 = b \\ x^3 + y^3 + z^3 = c \end{array} \right\}$ we shall, by Transposition, have

$$x + y = a - z$$

$$x^2 + y^2 = b - z^2$$

$$x^3 + y^3 = c - z^3.$$

Now, by multiplying together the two first of these Equations,

$$\text{we have } x^3 + x^2y + xy^2 + y^3 = ab - bz - az^2 + z^3.$$

And, by cubing of the First, we also have

$$x^3 + 3x^2y + 3xy^2 + y^3 = a^3 - 3a^2z + 3az^2 - z^3: \text{ Which,}$$

deducted from the Treble of the Former,

$$\text{leaves } 2x^3 + 2y^3 = 3ab - a^3 + 3a^2z - 3bz - 6az^2 + 4z^3:$$

And, This being $= 2c - 2z^3$ (by the third Equation) we therefore have

$$6z^3 - 6az^2 + \overline{3a^2 - 3b} \times z = a^3 - 3ab + 2c,$$

$$\text{and consequently } z^3 - az^2 + \frac{a^2 - b}{2} \times z = \frac{a^3 - 3ab + 2c}{6}.$$

From whence (when a , b and c are expressed in Numbers) three different Roots, or Values of z , may be found, answering all the Conditions of the Problem.

Thus, for Example, let $a=9$, $b=29$, and $c=99$; then our Equation will become $z^3 - 9z^2 + 26z - 24 = 0$.

And (by either of the two first Methods explained in Sect. 12. of my *Treatise of Algebra*) the three Roots, in this Case, will be found to be 2, 3 and 4.

Which Numbers are, therefore, the true Values of x , y and z , in the Equations $x+y+z=9$, $x^2+y^2+z^2=29$, and $x^3+y^3+z^3=99$.

Q U E S T I O N CVIII.

The Sum (a), the Sum of the Squares (b), the Sum of the Cubes (c), and the Sum of the Biquadrates (d), of any four Numbers being given; to determine the Numbers.

Let the four Numbers be denoted by x , y , z and u ; and put $A=a-u$, $B=b-u^2$, $C=c-u^3$, and $D=d-u^4$. From whence, by the Conditions of the Problem,

$$x + y + z = A,$$

$$x^2 + y^2 + z^2 = B,$$

$$x^3 + y^3 + z^3 = C,$$

$$x^4 + y^4 + z^4 = D.$$

Now, if the Second of these Equations, be subtracted from the Square of the First,

we shall have $2xy + 2xz + 2yz = A^2 - B$.

And if, in like Manner, the fourth Equation be subtracted from the Square of the Second, we shall have $2x^2y^2 + 2x^2z^2 + 2y^2z^2 = B^2 - D$.

Moreover, if from the Square of the former of these last Equations, the Double of the Latter be deducted, there will come out

$$8x^2yz + 8y^2xz + 8z^2xy = A^4 - 2A^2B - B^2 + 2D;$$

$$\text{or, } 8xyz \times x + y + z = A^4 - 2A^2B - B^2 + 2D;$$

$$\text{whence } xyz = \frac{A^4 - 2A^2B - B^2 + 2D}{8A} \quad (\text{because } x + y + z = A).$$

Again, by multiplying the first and, fifth Equations, into each other, we get $2x^2y + 2x^2z + 2y^2x + 2y^2z + 2z^2x + 2z^2y + 6xyz = A^3 - AB$.

And, by multiplying the First and Third together, there arises $x^3 + y^3 + z^3 + x^2y + x^2z + y^2x + y^2z + z^2x + z^2y = AB$:

The

The Double of which last taken from the Precedent, leaves
 $6xyz - 2x^3 - 2y^3 - 2z^3 = A^3 - 3AB$: And this, added to,
 $2x^3 + 2y^3 + 2z^3 = 2C$,
 gives $6xyz = A^3 - 3AB + 2C$.

Hence $\frac{A^3 - 3AB + 2C}{6} (= xyz) = \frac{A^4 - 2A^2B - B^2 + 2D}{8A}$

(*p. above*);

and consequently, by Reduction, $A^4 - 6A^2B + 8AC + 3B^2 - 6D = 0$.

In which Equation let the several Values of A, B, C and D, be now substituted, and then (dividing the Whole

by 24) we, at length, have $u^4 - au^3 + \frac{a^2 - b}{2} \times u^2 -$

$\frac{a^3 - 3ab + 2c}{6} \times u + \frac{a^4 - 6a^2b + 8ac + 3b^2 - 6d}{24} = 0$: Whose

four Roots (found by any of the known Methods) answer all the Conditions of the Problem.

QUESTION CIX.

To find the least Whole Number, which being divided by 19, shall produce a Remainder of 7; but, being divided by 28, the Remainder shall be 13.

Let $19x + 7$ denote the Number sought; where x , according to the Question, must be a whole Number. And, by the Question, it likewise appears that $19x + 7 - 13$, or, its Equal, $19x - 6$, must be divisible by 28 (without a Remainder).

But it is plain that $28x$ is divisible by 28: Therefore $(9x + 6)$ the Difference between $19x - 6$ and $28x$, must also be divisible by the same Number 28. For it is well known that, whatever Number, or Quantity, measures the Whole, and one Part, of Another (without a Remainder) must do the same by the remaining Part. Hence $(18x + 12)$ the Double of $9x + 6$, being divisible by 28, if the same be subtracted from $19x - 6$ (in order to get x without a Coefficient) the Remainder, $x - 18$, will, *still*, be divisible by the same Number; and consequently

frequently $x-18$, either, equal to Nothing, or to some Multiple of 28. But, as the least Value of x is required, $x-18$ must be $= 0$: And therefore $19x + 7 = 349$, the Number required.

QUESTION CX.

A certain Person bought as many Geese and Ducks, together, as cost him 14 Shillings; for the Geese he paid 2 s. 2 d. a-piece; and for the Ducks 1 s. 3 d. What Number had he of Each?

Let x denote the Number of the Geese, and y That of the Ducks;

so shall $26x + 15y = 168$, by the Question;

and therefore $y = \frac{168-26x}{15} = 11-x-\frac{11x-3}{15}$.

Which being a whole Number, by the Nature of the Problem, $11x-3$ must, therefore, be (exactly) divisible by 15.

But it is plain that $15x$ is divisible by 15; and that its Excess above $11x-3$, which is $4x+3$, must, *likewise*, be divisible by the same Number. Let the last Expression ($4x+3$) be now multiply'd by 3, and the preceding One ($11x-3$) subtracted from the Product (in order to get x without a Coefficient) whence you will have $x+12$; which being, *still*, divisible by 15, it is plain that x must either be 3, or 3 added to some Multiple of 15, as 18, 33, 48, &c. But it is apparent, from the Nature of the Question, that all these Numbers, except the First, are too large. Therefore there were 3 Geese, and 6 Ducks; which last Number (the Value of x being known) is found directly from the Equation exhibited above.

QUESTION

QUESTION CXI.

One having, at Play, won a certain Number of Guineas, not exceeding 100, and being asked to tell the Number, made this Reply: "If the Number of Guineas I have won be divided by 9, there will remain 6; but, if the Number of Shillings contained in them be divided by 39, there will remain 12." The Question is, to find what Number of Guineas he was a Winner of.

Let $9x+6$ denote the Number sought; where, according to the Question, x must be a whole Number. Then, the Number of Shillings being $189x+54$, it also appears that $\frac{189x+126-12}{39}$, or its Equal $4x+2+$

$\frac{11x+12}{13}$ must be a whole Number; and therefore $11x+12$, divisible by 13.

Let the Number 12 (for the sake of Brevity) be denoted by n :

Then, $13x$, and $11x+n$ being, both, divisible by 13, their Difference $2x-n$ must also be divisible by the same Number; and so, likewise, $\overline{2x-n} \times 5$, or its Equal $10x-5n$. And, if this be subtracted from $11x+n$, the Remainder $x+6n$ (or $x+72$) will, still, be divisible by

13. But $\frac{72}{13}$ is $= 5 + \frac{7}{13}$: Therefore $x+7$ must be

divisible by 13; and consequently the Value of x , either, equal to 6, or 6 added to some Multiple of 13: But, as the Value of $9x+6$ is not to exceed 100 (by the Question), That of x cannot be greater than 6: And therefore the Number sought can be no other than 60.

QUESTION

QUESTION CXII.

A Person, in exchange for a Number of Pieces of Foreign-Gold, valued at 17 s. 4 d. each, received a certain Number of Guineas (not exceeding 50) and one Shilling over. What was the Sum exchanged?

If x be put for the Number of Pieces of Foreign-Gold, and y for the Number of Guineas; then it is plain, from the Question, that $52x = 63y + 3$; and consequently that

$$x = \frac{63y + 3}{52} = y + \frac{11y + 3}{52}.$$

Here $11y + n$ (supposing $n = 3$) must be divisible by 52; as must, also, its Quintuple $55y + 5n$. And, if from this last, $52y$ be subtracted, and the Remainder be multiply'd by 4, we shall have $12y + 20n$; which must be, still, divisible by the same Number; and so likewise its Excess ($y + 19n$) above $11y + n$. But $\frac{19n}{52} =$

$\frac{57}{52} = 1 + \frac{5}{52}$; therefore $y + 5$ is, either, equal to 52, or to some Multiple of it; and consequently y equal, either, to 47, 99, 151, &c.

But, as the Value of y , by the Question, is not greater than 50, all the Numbers, after the First, are too large. Hence it appears that he received 47 Guineas and one Shilling, in exchange for 57 Foreign-Pieces; amounting in Value to 49 l. 8 s. Sterling.

QUESTION CXIII.

One laid out 10 Shillings in 20 Fowls, of three different Sorts, viz. Chickens, Pigeons, and Larks: The Chickens cost him 12 d, the Pigeons 4 d, and the Larks 1 d, a-piece. How many had he of Each?

Let x , y , and z denote the Numbers of the three several Sorts, respectively.

Then

Then will $\left\{ \begin{array}{l} x + y + z = 20 \\ 12x + 4y + z = 120 \end{array} \right\}$ by the Question.

And, by subtracting the former of these Equations from the Latter, we have $11x + 3y = 100$; and therefore $y =$

$$\frac{100 - 11x}{3} = 33 - 3x - \frac{2x - 1}{3}.$$

Now, $2x - 1$ being divisible by 3, it is evident that $(x + 1)$ its Difference from $3x$, must likewise be divisible by 3; and, consequently, that x must either be 2, or 2 increased by some Multiple of 3; that is, equal to some one of the Numbers 2, 5, 8, 11, 14. &c. But, as neither y nor z can be greater than 18 (by the Question) so all the foregoing Numbers, below and above 8, give the Value of y either too great, or too small.

But, when x is taken 8, y will come out $= 4$, and $z = 8$; which are the three Numbers required.

QUESTION CXIV.

To determine all the several Ways whereby it is possible to pay 60 l. in Guineas and Moidores, only.

Let x denote the Number of Guineas, and y the Number of Moidores.

Then will $21x + 27y = 1200$; or $7x + 9y = 400$, by the Problem; and therefore $x = \frac{400 - 9y}{7} = 57 - y -$

$\frac{2y - 1}{7}$. From whence, as $2y - 1$ is divisible by 7, it will

appear, by Reasoning as in the preceding Examples, that $y + 3$ must be divisible by the same Number; and consequently that the least Value of y is $= 4$,

and the corresponding Value of x $(= 57 - y - \frac{2y - 1}{7}) = 52$.

Now, having found the least Value of y , and the Greatest of x , the rest of the Answers will be obtained, by adding 7 (the Coefficient of x in the above Equation) to the last Value of y , continually, and subtracting 9 (the Coefficient of y) from the last Value of x . By means

means of which we get the 6 following Solutions, being all the Question admits of.

$$\text{viz. } \begin{cases} x = 52, 43, 34, 25, 16, \text{ or, } 7, \\ y = 4, 11, 18, 25, 32, \text{ or, } 39. \end{cases}$$

QUESTION CXV.

To find how many Ways it is possible to pay 20 l. in Half-Guineas; and Half-Crowns, without any other Sort of Coin.

If x be the Number of Half-Guineas, and y the Number of Half-Crowns; we shall have $21x + 5y = 800$; and therefore $y = \frac{800 - 21x}{5} = 160 - 4x - \frac{x}{5}$. Whence it appears, at one View, that x is a Multiple of 5: And therefore, the several, required, Values of x being expressed by 5, 10, 15, 20, 25, 30 and 35, Those of y , answering thereto, must be 139, 118, 97, 76, 55, 34, and 13, respectively. So that there are 7 Answers in this Case.

QUESTION CXVI.

A Reckoning of 20 Shillings was spent by a Company of twenty Persons, consisting of Officers, Sailors, and Marines: Each Officer paid 2 s. 6 d. each Sailor 12 d. and each Marine 8 d. How many Persons were there of each Denomination.

Let the three required Numbers be denoted by x , y , and z , respectively;

$$\text{so shall } \begin{cases} x + y + z = 20 \\ 30x + 12y + 8z = 240 \end{cases} \text{ by the Question.}$$

And, by subtracting 8 times the former of these Equations from the Latter, we shall have $22x + 4y = 80$;

$$\text{and therefore } y = 20 - \frac{11x}{2}.$$

But, y being a whole Number, it is plain that x must be an even Number, and, also, less than 4; and therefore

fore can be no other than 2. From whence y is given $= 9$, and $z = 9$, likewise.

QUESTION CXVII.

To find a Number, which, being divided by 28, shall produce a Remainder of 19; but, being divided by 19, the Remainder shall be 15; and, being divided by 15, the Remainder shall be 11.

Let $28x + 19$ denote the Number sought; where x , according to the first Condition of the Problem, must be a whole Number. And, by the second Condition, it appears that $28x + 19 - 15$ must be divisible by 19. Whence (following the Method observed in the preceding Examples) the least Value of x is found $= 8$: And so $19z + 8$ (where z denotes any whole Number) is a general Value of x , answering the two first Conditions.

Let this be, therefore, substituted instead of x ; and our assumed Expression will become $532z + 243$. From whence, as $532z + 232$ is divisible by 15, the least Value of z will be found $= 14$. And $15n + 14$, will be a general Value of z : Which, substituted in $532z + 243$, gives $7980n + 7691$ for a general Answer to the Problem; where n may be, either, equal to Nothing, or any whole Number.

QUESTION CXVIII.

To find three Numbers, in the proportion of 5, 7, and 9; which being, severally, divided by 11, 13, and 15, there shall remain 1, 2, and 3, respectively.

Let $5x$, $7x$, and $9x$ denote the three required Numbers: Then, by the Question, $5x - 1$, $7x - 2$, and $9x - 3$, must be, respectively, divisible by 11, 13, and 15 (without leaving any Remainder).

But it will be found (by proceeding as in the foregoing Problems) that the least Value of x , to answer the first of these Conditions, will be $= 9$: Therefore $9 + 11z$ (where

(where z denotes any whole Number) will be a general Value of x , answering the same Condition.

Let this Value be, therefore, substituted in the second and third Expressions; which, by that means, will become $77z + 61$, and $99z + 78$. And then, as the former of These is divisible by 13, the least Value of z , to fulfil this Condition, will (also) be found $= 9$.

Let, therefore, $9 + 13u$ (which is a general Value for z) be substituted in the last of the three Expressions, and it will become $13u + 9 \times 99 + 78$. Which being divisible by 15, the $\frac{1}{3}$ Part thereof, or $13u + 9 \times 33 + 26$ ($= 429u + 323$) must, consequently, be divisible by 5. Whence u is found $= 3$: Therefore z ($= 13u + 9$) $= 48$, and x ($= 11z + 9$) $= 537$. So that the three least Numbers, answering the Conditions of the Problem, are 2685, 3759, and 4833.

QUESTION CXIX.

Supposing $6x + 7y + 8z = 100$; 'tis required to find all the possible Values of x , y , and z , in whole Numbers.

In Questions of this Kind, where you have three, or more, indeterminate Quantities, and but one Equation, it will be proper, first of all, to find the Limits of those Quantities. Thus, because $x = \frac{100 - 7y - 8z}{6} = 16 - y - z - \frac{y + 2z - 4}{6}$, it appears that x cannot be greater than 14.

And, in the same Manner, it will appear that y cannot be greater than 12; nor z , greater than 10.

Now, as x is a whole Number, by the Question, $y + 2z - 4$ must therefore be divisible by 6: And, as $2z$ and 4 are even Numbers, it is plain that y must also be an even Number (since an odd One cannot be divided by an even One, without a Remainder). Let y be, therefore, first expounded by the least even Number (2), so will $y + 2z - 4$ become $= 2z - 2$; which, being divisible by 6, it is plain that $z - 1$ (the Half Thereof) is divisible

divisible by 3; and consequently that the several Values of z (when $y = 2$) are 1, 4, 7, and 10.

Whence the corresponding Values of x , by substituting above, will appear to be 13, 9, 5, and 1.

Let y be now taken $= 4$; then $y + 2z - 4$ will be $= 2z$: And so, z , being divisible by 3, the several Values of z , in this Case, will be 3, 6, 9. But the two first of These, only, are for our Purpose, the Last giving $x = 0$.

By taking $y = 6$, and proceeding in the same Manner, we shall get two other Answers; wherein z will be 2, and 5; and x , 7 and 3. And, by taking $y = 8$, two more Answers will be found (making 10 in the whole) which are all the Question admits of; and which, being placed in Order, will stand as below.

y	z	x
2	1. 4. 7. 10	13. 9. 5. 1
4	3. 6.	8. 4.
6	2. 5.	7. 3.
8	1. 4.	6. 2.

QUESTION CXX.

If $17x + 19y + 21z = 400$; 'tis proposed to find all the possible Values of x , y and z , in whole positive Numbers.

When the Coefficients of the indeterminate Quantities x , y and z are nearly equal, as in this Example, it will be convenient to substitute for the Sum of those Quantities: Thus, let $x + y + z = m$; then, by subtracting 17 times this last Equation from the preceding One, we shall have $2y + 4z = 400 - 17m$; and by subtracting the given Equation from 21 times the assumed One $x + y + z = m$, there will remain $4x + 2y = 21m - 400$. Therefore, since y and z can have no Values less than Unity, it is plain, from the first of these two Equations, that $400 - 17m$ cannot be less than 6, and therefore m

not

not greater than $\frac{400-6}{17}$ or 23: Also, because by the

second of the two last Equations, $21m-400$ cannot be less than 6, it is obvious that m cannot be less than

$\frac{400+6}{21}$ or 19: Therefore 19 and 23 are the Limits of

m in this Case. These being determined, let $4x$ be transposed in the last Equation, and the Whole divided

by 2, and we shall have $y = 10m - 200 - 2x + \frac{m}{2}$;

which being a whole Number, by the Question, it is evident that $\frac{m}{2}$ must likewise be a whole Number, and

consequently m equal to an even Number; which, as the Limits of m are 19 and 23, can only be 20, or 22:

Let, therefore, m be first taken $= 20$, then y will become $= 10 - 2x$ and $z(m-x-y) = 10 + x$; wherein x being taken equal to 1, 2, 3 and 4 successively, we shall have y equal to 8, 6, 4, 2 and z equal to 11, 12, 13, 14 respectively; which are four of the Answers required.

Again, let m be taken $= 22$, then will $y = 31 - 2x$ and $z = x - 9$, in which let x be taken equal to 10, 11, 12, 13, 14 and 15 successively, and y will come out $= 11, 9, 7, 5, 3$ and 1, and $z = 1, 2, 3, 4, 5$ and 6, respectively. Therefore we have the ten following Answers in whole Numbers; which are all the Question admits of.

$x =$	1	2	3	4	10	11	12	13	14	15
$y =$	8	6	4	2	11	9	7	5	3	1
$z =$	11	12	13	14	1	2	3	4	5	6

QUESTION

QUESTION CXXI.

To find two Whole Numbers, whereof the Difference of the Squares shall be 77.

Let the lesser Number be x , and the Greater $x+m$; and suppose the Number given to be represented by a :

So shall $(x+m)^2 - x^2 = a$,

that is, $2mx + mm = a$;

and consequently $x = \frac{a-mm}{2m} = \frac{a}{2m} - \frac{m}{2}$.

Whence we also have $x+m = \frac{a}{2m} + \frac{m}{2} (= \frac{a+mm}{2m})$.

But, in the Case proposed, a being $= 77$,

x becomes $= \frac{77}{2m} - \frac{m}{2}$, and $x+m = \frac{a}{2m} + \frac{m}{2}$:

Which being both required in Whole Numbers, it is evident, in the first Place, that m must be some

Divisor of 77; and, secondly, that $\frac{77}{m}$ must be greater than m ; and consequently m less than 9.

But the Divisors of 77, below 9, are 1 and 7: Which Numbers being wrote successively, in the Room of m , the corresponding Values of x will come out 33, and 2; and Those of $x+m$, 34 and 9, respectively: So that the Question, in the Case proposed, admits of two Answers, and no more.

QUESTION CXXII.

To find a Whole Number, to which 12 and 25 being, successively, added, both the Sums shall be square Numbers.

Let z be the Number sought; and assume x and $x+m$ for the Roots of the two Squares:

Then will $\left\{ \begin{array}{l} z+12 = x^2 \\ z+25 = (x+m)^2 \end{array} \right\}$ by the Question.

G

Hence,

Hence, by subtracting the former Equation from the Latter, we get $13 = \overline{x+m}^2 - x^2 = 2mx + mm$;

and therefore $x = \frac{13}{2m} - \frac{m}{2}$. Which being a Whole

Number, m must be $= 1$: Whence $x=6$; and $z (=x^2-12) = 24$.

QUESTION CXXIII.

To find three Whole Numbers, so that the Sum of the Squares of the Two least of them shall be equal to the Square of the Greatest.

It appears, from the Problem preceding the Last, that the Difference of the Squares of $\frac{a+mm}{2m}$, and $\frac{a-mm}{2m}$ is, universally, equal to a ,

or $\frac{\overline{a+mm}^2}{4mm} - \frac{\overline{a-mm}^2}{4mm} = a$; let a and m be what they will.

Whence it is also plain that $\overline{a+mm}^2 = \overline{a-mm}^2 + 4mma$.

But, since it is required to have $4mma$, a square Number (as well as the other Two) a must therefore be a square Number; let it be n^2 , and then our Equation will become $\overline{nn+mm}^2 = \overline{nn-mm}^2 = \overline{2mn}^2$: Where m and n may be expounded by any Whole Numbers, at Pleasure.

Thus, for Example, suppose $m=1$, and $n=2$; then there will come out $5^2 = 3^2 + 4^2$. Again, let $m=2$, and $n=3$, and there arises $13^2 = 5^2 + 12^2$. Lastly, let $m=2$ and $n=5$, and you will get $29^2 = 21^2 + 20^2$.

QUESTION

QUESTION CXXIV.

To find three Whole Numbers, whose Squares are in Arithmetical Progression.

Let x , $x+m$, and $x+n$ express three such Numbers. So shall $\overline{x+m}^2 - x^2 = \overline{x+n}^2 - \overline{x+m}^2$, by the Nature of the Problem.

Whence x is found $= \frac{nn-2mm}{4m-2n}$.

Put $4m-2n = a$, and $n^2-2m^2 = b$; then $x = \frac{b}{a}$, $x+m$

$= \frac{b+am}{a}$, and $x+n = \frac{b+an}{a}$. Now, as the Squares

of $\frac{b}{a}$, $\frac{b+am}{a}$, and $\frac{b+an}{a}$ are in Arithmetical Progression, it is plain that the Squares of their *Equimultiples*, b , $b+am$, and $b+an$, must be in Arithmetical Progression likewise. From whence, by expounding m and n by different Whole Numbers, successively, as many particular Answers as you please, may be exhibited.

Thus, if $m=2$ and $n=3$; then, a being $=2$, and $b=1$, there will come out 1, 5, and 7. But, if $m=3$ and $n=5$, we shall get 7, 13, and 17, for another Answer.

QUESTION CXXV.

Supposing $x^2 = z^2 + az + b$ (where a and b denote given Numbers); 'tis required to find the Values of x and z (if possible) in Whole Numbers.

Put $x = z + m$; then, by Substitution,

$$z^2 + 2mz + m^2 = z^2 + az + b;$$

and consequently $z = \frac{m^2-b}{a-2m}$. Which Value, by putting

$$a-2m=n \left(\text{or } m = \frac{a-n}{2} \right) \text{ becomes } = \frac{a^2-2an+n^2-4b}{4n} =$$

$\frac{1}{4}$ of $\frac{aa-4b}{n} = 2a+n$. From which it is evident, that, to have the Value of z a Whole Number, n must be some Divisor of the given Quantity $aa-4b$, and therefore even or odd, according as a is even or odd.

Example. Suppose the given Equation to become $x^2 = z^2 + 20z$; in which Case, a being $= 20$, and $b = 0$, we have $z = \frac{1}{4} \times \frac{400}{n} = 40 + n$; where the, even, Divisors of 400 are 2, 4, 8, 10, &c. whereof the Second will be found to answer; the Values of z and x coming out 16 and 24, respectively.

Again, suppose the given Equation to become $x^2 = z^2 + 100z + 1000$: Here we have $z = \frac{1}{4} \times \frac{6000}{n} = 200 + n$: And the, even, Divisors of 6000, are 2, 4, 6, 8, 10, 12, 16, 20, &c. Whereof 4, 12, and 20 succeed: By the last of These (which determines the least Values) z comes out $= 30$, and $x = 70$.

QUESTION CXXVI.

Having given $x^2 = a^2 + bz + cz^2$, wherein a , b and c denote given Whole Numbers; 'tis required to find the Values of x and z (if possible) in Whole Numbers.

Put $x = a + mz$; then will $a + mz^2 = a^2 + bz + cz^2$; that is, $a^2 + 2amz + m^2z^2 = a^2 + bz + cz^2$.

Whence z comes out $= \frac{b-2am}{mm-c}$:

Where it is evident, that, in order to have a positive Value, m must be taken equal to some Number between

\sqrt{c} , and $\frac{b}{2a}$.

Thus, supposing the given Equation to become $x^2 = 64 - 12z + 5z^2$, the Value of m , in this Case, must be less

less than $\sqrt{5}$, and greater than $-\frac{12}{16}$. Let it therefore be expounded by 2 and 1, successively;

Whence $\frac{-12-16m}{mm-5}$, or its Equal $\frac{12+16m}{5-mm}$ (which is, here, the Value of z) will come out 44 and 7 respectively; and the corresponding Values of x ($a+mx$) are found to be 96 and 15: Both which answer the Conditions of the Problem.



PART II.

CONTAINING

A Variety of GEOMETRICAL PROBLEMS,
with their SOLUTIONS: Both by ALGEBRA,
and independent of it, from Principles purely
GEOMETRICAL.

ALTHOUGH the chief Design of this Work consists not in Exhibiting a Number of Rules and Precepts, but in illustrating Those already known, on the Subject, by a Set of proper and useful Examples; yet, as we are in this Part to treat of the Investigation of Geometrical Problems, without calling in the Assistance of Algebra (a Thing hitherto very little considered by Authors, tho' in itself very interesting and useful) it may not be amiss, before we proceed to particular Cases, to premise a few General Observations on this Head.

I. In the first Place, then, it is necessary, in order to the Construction of Geometrical Problems, that something of the *Geometric Loci* should be understood—Thus it will be of Use to know that the Place of the Vertex of a Triangle, whereof the Base and opposite Angle are supposed to remain constant, while the other Sides and Angles vary, will *always* fall in the Circumference of a Circle passing thro' the Extremities of the Base. This appears from *Euclid B. 3. Prop. 21*; and is also demonstrated in my *Elements of plane Geometry, B. 3, Prop. 9.*

It ought moreover to be known, that, the Locus of the Vertex, when the Ratio of the two Sides and the
Base

Base of the Triangle are given, or continue invariable, will; *also*, be the Circumference of a Circle, dividing the Base in the given Ratio (*For the Demonstration of which, see Elem. plane Geometry B. 4, Prop. 15.*).

If the Sum, or the Difference, of the Squares of the two Sides, together with the Base, be supposed given, the Locus of the Vertex will, *still*, be the Circumference of a Circle, in the former Case; and a Right-line, in the Latter (*Vid. B. 2, Prop. 12.*).

But, if the Sum, or the Difference, of the Sides, themselves, is given; then the Locus will, either, be the Arch of an Ellipsis or an Hyperbola (as is well known to Those who have touch'd upon Conic-Sections). But these two Last are not admitted in the Construction of linear and plane Problems; of which, only, we purpose to treat — The Problems, of the following Collection; wherein the Use of the *Geometric-Loci*, above specified, is particularly exemplified, are the 11, 12, 19, 34, 35, 48, and 51st.

2°. When, in the Figure to be constructed, the Sum, or the Difference, of two adjacent Sides happens to be given; it will be proper, first, to form a Triangle, so that the said Sum, or Difference, may be one of its Sides; and, then, to consider, what other Sides, or Angles, will be given, or become known, in Consequence Thereof. This Rule is illustrated in the 1, 2, 18, 26, 27th, and some other of the succeeding Problems.

3°. It often happens that the Ratio of two, or more, Lines is given, from the Nature of the Figure, or by Hypothesis, though the Lines themselves are absolutely unknown: In all such Cases we must endeavour, by drawing Parallels (or some other Way) to obtain other Lines in the same given Ratio; so that, one of them being given, or known from the Nature of the Figure or Problem, the other may also become known — The Use of this Rule, which is very extensive, will particularly appear in the Solutions to the 3, 13, 16, 22, 26, 33, 37, and 56th. Problems.

4°. But, if the Lines, whereof the Ratio is given, should happen to lye remote of each other; then, by the Help of Those, we must endeavour to determine the Ratio of Others, lying nearer together; and so on, till we obtain (if possible) the Ratio of two Lines, that are both Sides of the same Triangle; wherein one Angle and the remaining Side (or some other, two, Parts) are given — For the better understanding this Rule, consult the Solutions to the 49th and 57th Problems, in particular.

5°. Lastly, when the Rectangle under two unknown Lines is given, either, a mean Proportional must be found, or else, two other Lines must be assigned, by forming similar Triangles (or some other Way) so as to comprehend an equal Rectangle; and so that, one of them being given by the Nature of the Figure, the Other may also become known. This Rule is exemplified in the 4, 5, 6, 9, 10, 21, 38, 40, 41, 44, and 53^d. Problems.

Besides the above, other Observations might be here laid down; but Those already delivered being the most General that have occur'd to me, I shall now proceed on in the Resolution of Problems; the proper Business of this Work.

P R O B L E M

PROBLEM I.

One Leg BC, and the Difference between the other Leg AB and the Hypotenuse AC, of a right-angled Triangle ABC, being given; to find both AB and BC.

Put $BC=a$, $AB=x$, and $AC=x+b$; (b being the given Difference) Then,

\overline{AC}^2 being $= \overline{AB}^2 +$

\overline{BC}^2 (*Elem.* 7. 2. *) we

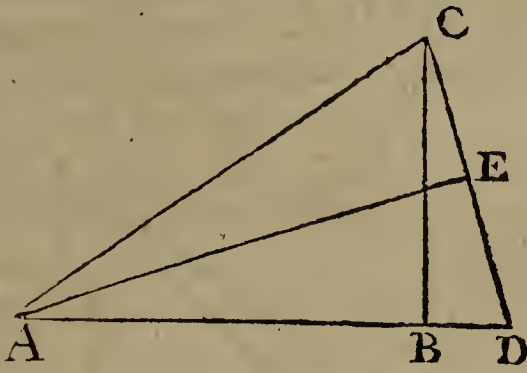
have $xx + 2bx + bb =$

$xx + aa$: Whence $2bx$

$= aa - bb$; and conse-

quently $x = \frac{aa-bb}{2b} = \frac{aa}{2b}$

$-\frac{b}{2}$. From which AC ($x+b$) is given $= \frac{aa}{2b} + \frac{b}{2}$.



Geometrically.

If, in AB produced, there be taken BD equal to the given Difference of AC and AB, and DC be drawn (according to the second General Observation) it is evident that AD will be equal to AC; and the Angle ACD, also, equal to the Angle D.

Therefore, having taken BD as above specified, and made BC perpendicular thereto, and of the given Length, and joined D, C; let CA be so drawn as to make the Angle DCA=D; or, instead thereof, let a perpendicular EA be erected on the middle of CD; then the Intersection of either of these Lines with DB, produced, determines the Triangle.

From this Construction we have the very same Theorem, for the numerical Solution, as is derived above, from the Algebraical Process: For the Triangles ADE and CDB, having each a Right-Angle, and D

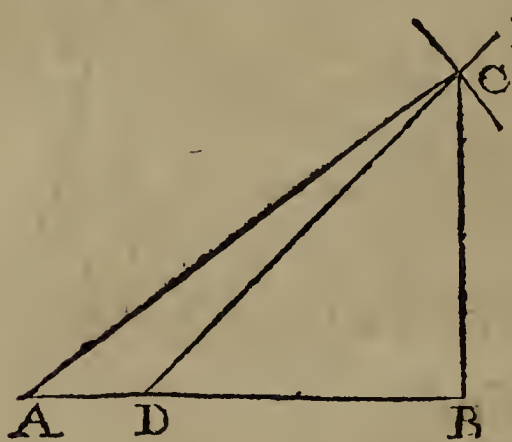
* Note. The Quotations, in this, and the succeeding Problems, refer to my *Elements of Plane Geometry*; printed for J. NOURSE.

common,

common, are similar. Therefore $DB : DC :: DE$
 $(\frac{1}{2}DC) : AD (AC) = \frac{DC^2}{2DB} = \frac{BC^2 + BD^2}{2BD} = \frac{BC^2}{2BD} + \frac{BD}{2};$
as before.

P R O B L E M II.

The Hypotenuse AC, and the Difference of the two Legs AB and BC, of a right-angled Triangle ABC, being given; to determine the Legs.



Put $AC = a$, $AB = x$,
 and $BC = x - b$:

So shall $xx + x - b]^2 = aa$
 (*Elem. 7. 2.*).

that is, $2xx - 2bx + bb = aa$.

Whence $xx - bx = \frac{aa}{2} - \frac{bb}{2}$:

And consequently $x =$
 $\sqrt{\frac{1}{2} aa - \frac{1}{4} bb} + \frac{1}{2} b$.

Geometrically.

If, in AB there be taken AD equal to the given Difference, and CD be drawn; then, DB being = BC, the Angle BDC will also be = BCD = Half a Right-Angle.

Therefore, having laid down AD, and drawn an indefinite Line DCE to make the Angle BDE = $\frac{1}{2}$ Right-Angle; upon the Center A, with the given Interval AC, let an Arch be described, intersecting DE in C; from which Point, upon AD produced, let fall the Perpendicular CB; so shall ABC be the Triangle required.

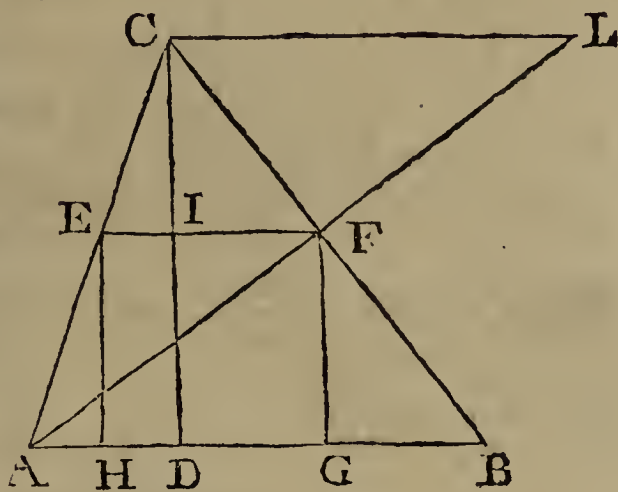
The numerical Solution, according to this Construction, is very easy, by the Help of Trigonometry: For, two Sides and one Angle of the Triangle ADC being given, the other Angles may from thence be found; and then, all the Angles and one Side (AC) of the proposed Triangle being known, the other Sides AB and BC may also be determined.

PROBLEM

PROBLEM III.

The Base AB and the Perpendicular CD of any Triangle ABC being given; to find the Side EF, or EH, of the inscribed Square EFGH.

Put $CD=a$, $AB=b$, and $DI(=EF)=x$: Then will $CI=a-x$; and, by Reason of the similar Triangles ABC and ECF , we shall have $a:b::a-x:x(=EF)$. Therefore $ax=ab-bx$; and consequently $x=\frac{ab}{a+b}$.



Geometrically.

The Ratio of EH to EA being given, as CD to CA, EF must therefore be to EA in the same given Ratio: And, if CL be drawn parallel to EF, meeting AF produced in L (*agreeable to the 3^d General Observation*) it is evident, because of the similar Triangles, that the Line CL, so drawn, will be to CA, *still*, in the same given Ratio; that is, $CL:CA::CD:CA$; and consequently $CL=CD$. Whence the Method of Construction is manifest.

PROBLEM IV.

To determine the Sides of a Rectangle, EFGH, inscribed in a given Triangle ABC, whose Area shall be to That of the Triangle in a given Ratio.

Put the Perpendicular $CD=a$, the Base $AB=b$, and the Altitude EH of the Rectangle $=x$; and let the given Ratio of ABC to $EFGH$ be that of m to n .

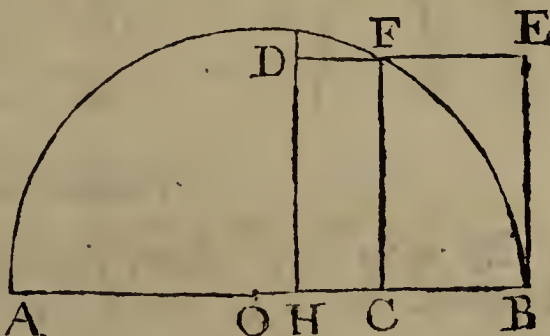
Because

This Problem, it is observable, becomes impossible when MN passes intirely without the lesser Semi-circle, that is, when the given Rectangle is supposed greater than half the Triangle. The same Thing appears also from the Algebraic Solution, in which Case the Quantity $\left(\frac{aa}{4} - \frac{naa}{2m}\right)$ under the Radical-Sign, becomes negative.

P R O B L E M V.

To divide a given Right-line AB into two such Parts AC and BC, that the Rectangle contained under them may be of a given Magnitude.

Put $AB=a$, and $AC=x$, and let the given Magnitude, or Content, of the proposed Rectangle be represented by the Square BD whose Side BE, or ED, let be denoted by b : Then will $x \times a - x = bb$; or $xx - ax = -bb$.



Whence $x = \pm \sqrt{\frac{1}{4} aa - bb} + \frac{a}{2}$.

Geometrically.

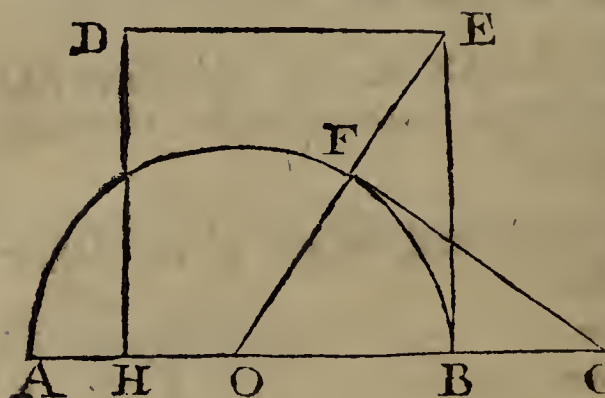
If upon AB, as a Diameter, a Semi-circle AFB be described, it is evident (*by Elem. II. 4.*) that a Perpendicular FC, drawn from the Point wherein the Circumference intersects DE, will cut AB in the Point required—It is plain, from both these Solutions, that the given Rectangle must not be greater than the Square of half the given Line, to be divided.

PROBLEM

PROBLEM VI.

To a given Line AB , it is required to add another Line BC , so that the Rectangle under the whole, compounded, Line AC , and the Part added, may be of a given Magnitude.

Let $AB=a$, $BC=x$, and the Side of the given



Square $BEDH$, expressing the Magnitude of the proposed Rectangle, $= b$.

Then we shall have

$$\overline{a+x} \times x = bb; \text{ and consequently } x =$$

$$\sqrt{bb + \frac{1}{4}aa} - \frac{1}{2}a.$$

Geometrically.

If, upon AB , a Semi-circle be described, and CF be supposed drawn to touch it in D , it is plain, *from Elem.*

Corol. to 17. 3, that \overline{CF}^2 is $= AC \times BC = \overline{BE}^2$ (*by Hypothesis*); and consequently $CF = BE$: Therefore, OF being $= OB$, it follows that OE and OC are likewise equal. Hence, if to the Middle of AB , we draw EO , and take $OC = EO$, the Thing is done.

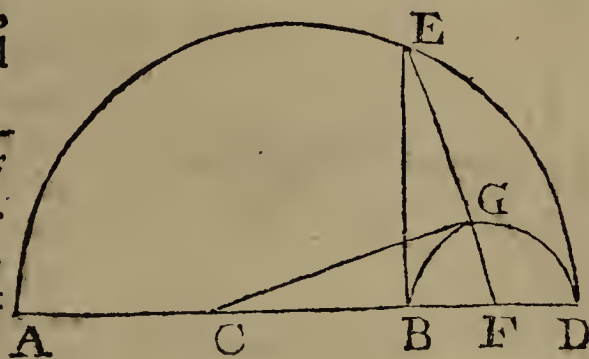
PROBLEM

PROBLEM VII.

To divide a given Right-line AB into two such Parts, that, the Rectangle under one of them AC and another, given, Line BD, may be equal to the Square of the remaining Part BC.

Put $AB=a$, $BD=b$,
and $BC=x$; then will
 $AC = a-x$,

and therefore $xx = a-x$
 $\times b$, by the Question.
Hence $xx + bx = ab$;
and, consequently $x =$
 $\sqrt{ab + \frac{1}{4}bb - \frac{1}{2}b}$.



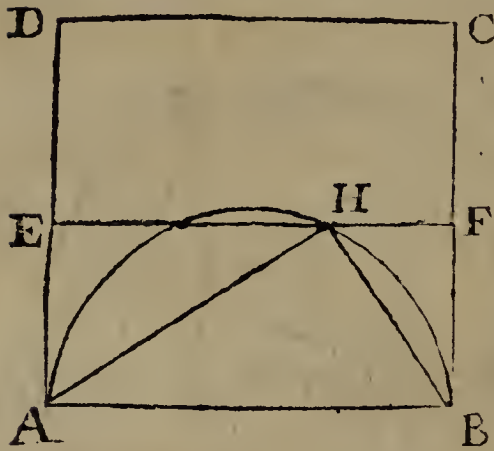
Geometrically.

Since $\overline{BC}^2 = AC \times BD$ (by Hypothesis) it follows, by adding $BC \times BD$ to Each, that $\overline{BC}^2 + BC \times BD = AC \times BD + BC \times BD$; or that, $BC \times CD = BD \times AB = \overline{BE}^2$, taking BE a Mean Proportional between BD and AB (by Elem. 14. 5.) But the Rectangle $BC \times CD$, if a Semi-circle be described upon the Diameter BD, is known to be equal to the Square of the Tangent CG (Elem. Cor. to 17. 3). Hence $\overline{CG}^2 = \overline{BE}^2$; and consequently $CG = BE$: Therefore, FG being also = FB, it follows that FC is equal to FE; whence the Method of Construction is manifest.

PROBLEM

P R O B L E M VIII.

To determine two Lines, whereof the Rectangle shall be equal to a given Rectangle ABFE, and the Sum of their Squares equal to a given Square ABCD.



Put $AB (=BC) = a$, $BF (=AE) = b$, and let the two required Lines be denoted by x and y .

Then will $xy = ab$, and $xx + yy = aa$, by the Question: Whence by adding, and subtracting, the Double of the former of these Equations from the Latter,

we have
$$\begin{cases} xx + 2xy + yy (= \overline{x+y}^2) = aa + 2ab, \\ xx - 2xy + yy (= \overline{x-y}^2) = aa - 2ab. \end{cases}$$

Therefore
$$\begin{cases} x+y = \sqrt{aa+2ab} \\ x-y = \sqrt{aa-2ab} \end{cases}$$

From which
$$\begin{cases} 2x = \sqrt{aa+2ab} + \sqrt{aa-2ab} \\ 2y = \sqrt{aa+2ab} - \sqrt{aa-2ab}. \end{cases}$$

Geometrically.

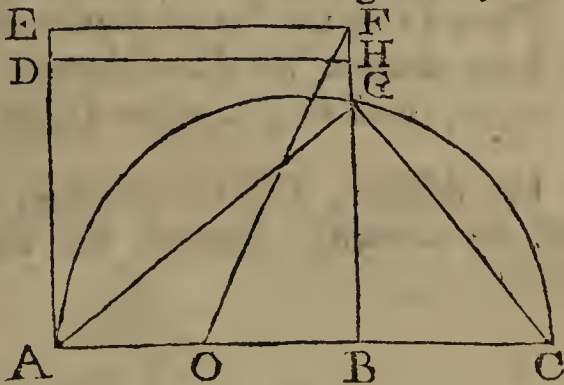
If upon AB a Semi-circle be described, intersecting EF in H , then the Lines joining A, H , and B, H will answer the Conditions of the Problem. For, the Angle AHB being a Right-one (*Elem.* 11. 3.) thence is $\overline{AH}^2 + \overline{BH}^2 = \overline{AB}^2 = ABCD$ (*Elem.* 7. 2.); and $AH \times BH (= \text{twice the Triangle } ABH) = ABFE$ (*Elem. Corol. to 2. 2.*).

PROBLEM

P R O B L E M IX.

To determine two Lines, whereof the Rectangle shall be equal to a given Rectangle ABFE, and the Difference of their Squares equal to a given Square ABHD.

Put $AB=a$, $BF=b$; and let x be the greater, and y the lesser, of the two Lines required: So shall $xy=ab$; and $xx-yy=aa$, by the Question. From the former of which Equations we have $y=\frac{ab}{x}$; which



Value, substituted in the Latter, gives $xx - \frac{aabb}{xx} = aa$.

Hence $x^4 - a^2x^2 = a^2b^2$;

and consequently $x = \sqrt{\frac{1}{2}aa + a\sqrt{bb + \frac{1}{4}aa}}$:
Whence y will also be known.

Geometrically.

It is evident, in the first Place, that the two Lines to be determined will be the Hypothenufe and one Leg of a right-angled Triangle (ABG) whose remaining Leg is the given Line AB. And, since the Rectangle under these Lines is supposed given, another Triangle ACG, similar to ABG, must therefore be assumed; so that CG, in the Former, and BG, in the Latter, may be homologous Sides (according to the 5th General Observation; vid. p. 88.)

Hence we have $CG \times AB = BG \times AG$ (Elem. 18. 3) $= AB \times BF$ (by Hyp.) and consequently $CG = BF$;

And so, $AC \times BC (=CG^2, \text{Elem. Cor. 11. 4})$ being given $= BF^2$, the Case under Consideration is now reduced to our 6th Problem: Whence we have the following Construction.

H

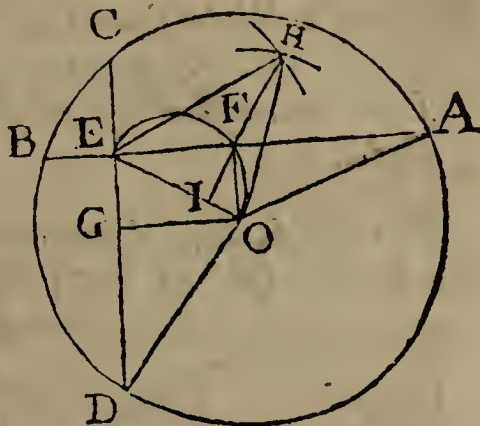
To

the Perpendiculars OF and OG; and draw the Radii OA and OD: Also put AF ($= \frac{1}{2} AB$) = a , DG ($= \frac{1}{2} CD$) = b , OE = c , and AO ($= DO$) = x .

Then will $\overline{OF}^2 = x^2 - a^2$,
and $\overline{OG}^2 = x^2 - b^2$; but

$\overline{OF}^2 + \overline{OG}^2 (= \overline{OF}^2 + \overline{FE}^2)$ is $= \overline{OE}^2$; that is, $2x^2 - a^2 - b^2 = c^2$; and conse-

quently $x = \sqrt{\frac{aa+bb+cc.}{2}}$



Whence the Diameter is given $= \sqrt{2a^2 + 2b^2 + 2c^2}$.

Geometrically.

Since $\overline{FE}^2 + \overline{DG}^2$ is $(= \overline{OD}^2 = \overline{OA}^2)$
 $= \overline{OF}^2 + \overline{AF}^2$, it is evident that $\overline{FE}^2 - \overline{OF}^2$ is
given $= \overline{AF}^2 - \overline{DG}^2$. We are therefore to con-
struct a right-angled Triangle upon the given Hypothe-
nuse OE, whereof the Squares of the two Legs shall
have the same Difference as the two given Squares \overline{AF}^2
and \overline{DG}^2 .

In order to which, upon OE, let a Semi-circle be described; also, from the Centers O and E, with Radii equal to DG and AF, respectively, let two Arcs be described, intersecting each other in H; from which Point, upon OE, let fall the Perpendicular HG; which will intersect the Semi-circle in (F) the Vertex of the required Triangle: Since it is evident that $\overline{FE}^2 - \overline{OF}^2 = \overline{EI}^2 - \overline{OI}^2 = \overline{EH}^2 - \overline{OH}^2 = \overline{AF}^2 - \overline{DG}^2$ (by Construction.)

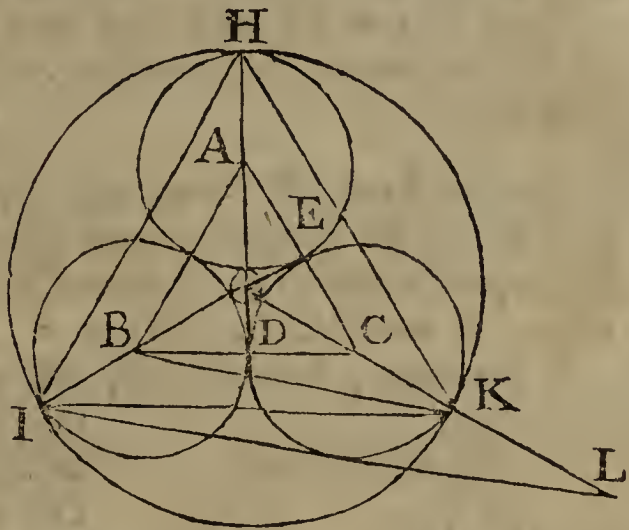
Therefore, if in EF produced, FA be taken of the given Length, OA (when drawn) will be the Radius of the required Circle.

cannot be given less than That of OA to OC, without rendering the Problem impossible.

PROBLEM XIII.

To determine the Radii of three equal Circles A, B, C, described in a given Circle HIK to touch each other and likewise the Circumference of the given Circle.

Let the Centers of the several Circles be joined; and let AO and BO be produced to bisect BC and AC in D and E: Also let the Radius (OI) of the given Circle be denoted by a , and That of each of the required Ones by x . Now the Triangles BCE and BOD being similar, and $CE = \frac{1}{2}$



BC, it appears that OD is also $= \frac{1}{2} OB$. But $OB^2 - OD^2$ is $= BD^2$; that is, in Species, $a^2 - x^2 - \frac{(a-x)^2}{4} = xx$. Which, solved, gives $x = \sqrt{12aa - 3a} = a \times 2\sqrt{3} - 3$.

Geometrically.

It is evident that the Right-line IK, joining the Points of Contact I and K, is the Side of an Equilateral Triangle inscribed in the given Circle: And, that, if in OK produced there be taken $KL = \frac{1}{2} IK$, a Line drawn from I to L, will be parallel to another Line drawn from B to K; because the Triangles IKL and BCK (having $IKL = BCK$, and $IK : KL :: 2 : 1 :: BC : CK$) are equiangular.

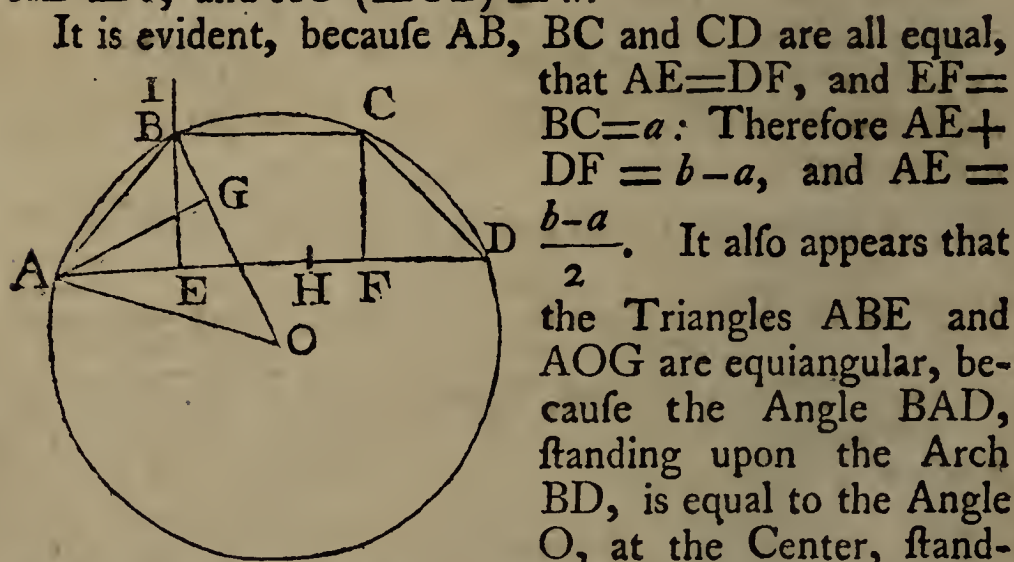
Therefore, in order to the Geometrical Construction, having first drawn the Radii OH, OI, and OK, to divide

divide the Circumference in three equal Parts, and taken KL, in OK produced, equal to $\frac{1}{2}$ IK; draw LI, and KB, parallel thereto, meeting OI in B; make OA and OC each = OB; and upon the Centers A, B, and C, through A, I, K let the three required Circles be described.

P R O B L E M XIV.

Supposing AB and ABD to be two Arcs of a Circle, in the Ratio of 1 to 3; and that both their Chords AB and AD are given; 'tis required to find the Radius of the Circle.

Let the Arch BD be equally divided in C, and upon AD let fall the Perpendiculars BE and CF; also let AG be perpendicular to the Radius OB: And put $AB = a$, $AD = b$, and $AO (=OB) = x$.



It is evident, because AB, BC and CD are all equal, that $AE = DF$, and $EF = BC = a$: Therefore $AE + DF = b - a$, and $AE = \frac{b-a}{2}$. It also appears that the Triangles ABE and AOG are equiangular, because the Angle BAD, standing upon the Arch BD, is equal to the Angle O, at the Center, standing upon AB ($= \frac{1}{2}$ BD): Hence we have, $AB (a) :$

$$AE \left(\frac{b-a}{2} \right) :: AO (x) : OG = \frac{\overline{b-a} \times x}{2a}. \quad \text{But}$$

$$\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2 - 2 \, OB \times OG; \quad \text{that is, in}$$

$$\text{Species, } a^2 = x^2 + x^2 - \frac{\overline{b-a} \times x^2}{a}; \quad \text{or } a^3 = 3ax^2 - bx^2.$$

$$\text{Therefore } x = \sqrt{\frac{a^3}{3a-b}} = a \sqrt{\frac{a}{3a-b}}.$$

From

From the same Equation, if the Radius AO (x) and the Chord (b) of an Arch ABD be supposed given, the Chord AB (a) of the Sub-triple of that Arch, may be determined: But this, by-the-bye.

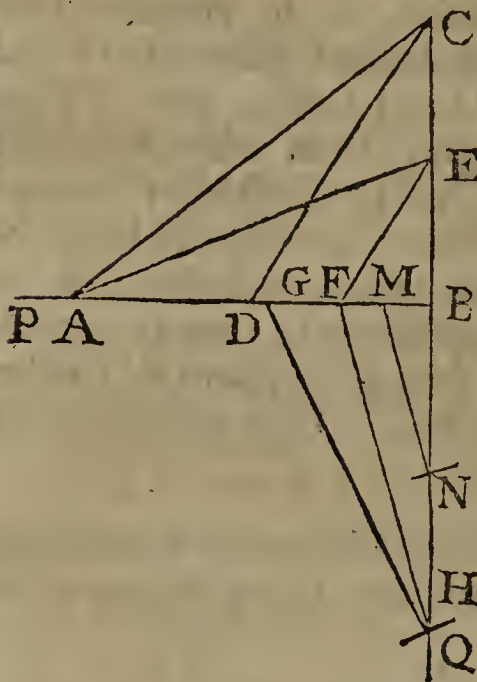
Geometrically.

The Geometrical Construction of the proposed Problem is also obvious from the known Value of AE and the Equality of the Angles O and EAB ; and is thus. Draw AD of the given Length, from which take $DH = AB$; let the Remainder AH be bisected by the perpendicular EI ; to which draw AB so as to be of the given Length; and upon the same, as a Base, let an Isosceles Triangle AOB be constituted, whose vertical Angle O shall be $= EAB$; then it is evident that either of the equal Sides AO , or BO , will be the Radius of the Circle.

As to the Trigonometrical Calculation, it is too plain, from the Construction, to need any Thing further to be said about it.

PROBLEM XV.

Having the Lengths of two Lines AE and CD drawn from the acute Angles of a right-angled Triangle ABC, to bisect, and terminate in, the opposite Sides; to determine the Triangle.



Call AE, a ; CD, b ; and BD (or AD), x ; Then will $\overline{CB}^2 = b^2 - x^2$; whence $\overline{BE}^2 (= \frac{1}{4} \overline{CB}^2) = \frac{bb - xx}{4}$; and therefore $a^2 (= \overline{AE}^2 = \overline{BE}^2 + \overline{BA}^2) = \frac{bb - xx}{4} + 4xx$.

Hence $15xx = 4aa - bb$;

and $x = \sqrt{\frac{4aa - bb}{15}}$.

Geometrically.

If EF be supposed parallel to CD, meeting AB in F, the Length thereof, being half That of CD, will consequently be given: Whence $\overline{AB}^2 - \overline{BF}^2 (= \overline{AE}^2 - \overline{EF}^2)$ is also given.

Therefore the Problem is reduced to this; To determine two Lines AB and BF, in the Ratio of 4 to 1, so that the Difference of their Squares may be equal to the Difference of the Squares of two given Lines AE and EF.

Hence, having drawn two indefinite Lines BP and BQ at right-Angles to each other, take BG, in the Former, equal to EF; and from the Point G, to BQ, draw

draw $GH=AE$: So shall $\overline{AB}^2 - \overline{BF}^2 (= \overline{AE}^2 - \overline{EF}^2) = \overline{BH}^2$ (*Elem. Cor. 7. 2*). Therefore, if from any Point M in PB , to BQ there be drawn $MN = \frac{1}{4} MB$; it is evident that a Line, HF , drawn from H parallel to NM , will cut off BF as required—This Problem becomes impossible when either of the two given Lines is greater than the Double of the Other.

P R O B L E M XVI.

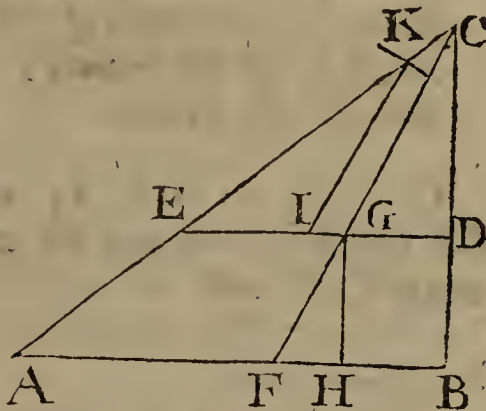
The Length and Position of a Right-line DE, drawn parallel to the Base of a given right-angled Triangle ABC being known; 'tis proposed to draw another Right-line CF from the Vertex of the Triangle, so that the Part thereof (FG) intercepted by AB and ED, may be equal to (EG) the Part of ED intercepted by CA and CF.

Upon AB let fall the Perpendicular GH :

And put $ED=a$, $CD=b$, $DB=c$, and $EG=x$:

Then, from the Similarity of the Triangles CDG and GHE , it will be, $CD (b) : DG (a-x) :: GH (c) :$

$$HF = \frac{c \times a - x}{b} :$$



$$\text{And therefore } \frac{c^2 \times a - x^2}{b^2} + c^2 (HF^2 + GH^2) = x^2 (GF^2)$$

$$\text{Whence, by Reduction, } b^2 x^2 - c^2 x^2 + 2c^2 ax = c^2 a^2 + c^2 b^2 ;$$

$$\text{and } x^2 + \frac{2ac^2}{bb - cc} \times x = \frac{a^2 + b^2 \times c^2}{bb - cc} .$$

$$\text{From which } x \text{ is found } = \frac{bc \sqrt{aa + bb - cc} - ac^2}{bb - cc} .$$

Geometrically.

Geometrically.

Since GF is to GC every-where in the given Ratio of DB to DC, GE, in the required Position, must therefore be to GC, in the same, given, Ratio. Hence, if, in ED, there be taken EI = DB, and from the Center I, at the Distance of CD, an Arch be described, intersecting EC in K; then a Line CGF, drawn parallel to the Radius KI, will determine both the Length and Position of GF: For it is evident that $CG : EG :: EK : EI :: CD : DB :: CG : GF$; and consequently that $EG = GF$ —This Problem appears to be impossible when BD is greater than CE.

P R O B L E M XVII.

Supposing the Area of a Square BE DF, form'd within a given right-angled Triangle ABC, to be equal to the Area of the Triangle ADC, made by drawing Lines from the Extremes of the Hypothenuse to the adjacent Angle D of the Square; 'tis proposed to determine the Side of the Square.

Call BC, a ; BA, b ; and BE (or BF), x : Then, CE being $= a - x$ and AF $= b - x$, the Area of the Triangle CED will

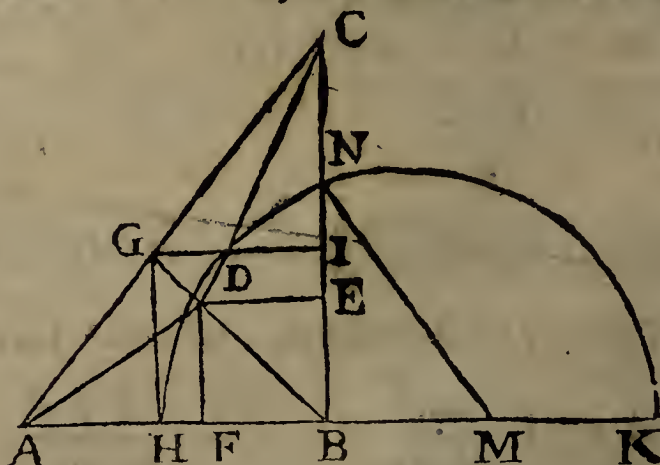
$$be = \frac{x \times a - x}{2},$$

and That of the Triangle AFD

$$= \frac{x \times b - x}{2}.$$

But it is evident that BEDF + CED + AFD + ADC is = ABC: Which, because ADC is equal to BEDF, also gives $2BEDF + CED +$

$$AFD = ABC, \text{ that is, } 2x^2 + \frac{x \times a - x}{2} + \frac{x \times b - x}{2} =$$



$= \frac{ab}{2}$: Hence $x^2 + \frac{a+b}{2} \times x = \frac{ab}{2}$; and $x =$

$$\sqrt{\frac{ab}{2} + \left[\frac{1}{4}a + \frac{1}{4}b\right]^2} - \frac{a+b}{4}. \quad \text{Q. E. I.}$$

Geometrically.

If BD be produced to meet AC in G, and GH be drawn perpendicular to AB, it is evident that the Triangle ADC will be to the Triangle ABC $\left(\frac{AB \times BC}{2}\right)$ as GD : GB, or as HF to HB (*Elem. 2. and 5. of the 4*): And it also appears (*by Elem. 14. 4.*) that $HB \times \overline{AB+BC}$ is $= AB \times BC$. Therefore it follows that
 Triang. ADC : $\frac{HB \times \overline{AB+BC}}{2} :: HF : HB :: HF \times \frac{AB+BC}{2} : HB \times \frac{AB+BC}{2}$; and consequently that the
 Triangle ADC $= HF \times \frac{AB+BC}{2} = HF \times BK$; by
 taking BK (in AB produced) equal to $\frac{AB+BC}{2}$.

Hence, \overline{BF}^2 being $(=ADC) = HF \times BK$, the Case under Consideration is reduced to our 7th Problem; and the Geometrical Construction will, therefore, be as follows.

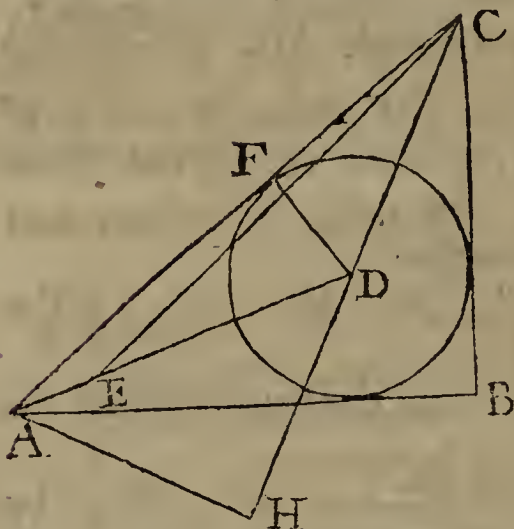
Having drawn BG (to bisect the Angle ABC), and GH perpendicular to AB, and also taken BK equal to Half the Sum of AB and BC (*as above intimated*), let a Semi-circle, upon HK, be next described, intersecting BC in N; from which Point, to the Middle of BK, let NM be drawn; then make MF = MN, and BF will be the Side of the Square; as is manifest from the Problem above quoted.

PROBLEM

P R O B L E M XVIII.

Having given the Hypotenuse AC of a right-angled Triangle ABC, and the Difference of two Lines AD and CD, drawn from the Extremes thereof to the Center D of the inscribed Circle; to determine the remaining Sides AB and BC, of the Triangle.

Upon CD, produced, let fall the Perpendicular AH;



and make $AC = a$, $AD = x$, $DC = y$, and the given Difference, $x - y$, $= b$.

It is evident that the Angle ADH is $= DAC + DCA = \frac{1}{2} BAC + \frac{1}{2} BCA = \frac{1}{2}$ a Right Angle; and therefore $AH = HD$

$= \frac{AD}{\sqrt{2}} = \frac{x}{\sqrt{2}}$. But

$CD^2 + AD^2 + 2DH \times CD = AC^2$; that is, in Species, $y^2 + x^2 + xy\sqrt{2} = a^2$. Which Equation, by substituting $x - b$ instead of y its Equal, and denoting $\sqrt{2}$ by c , becomes $x^2 - 2bx + b^2 + x^2 + cx^2 - cbx = a^2$; that is $\frac{2+c}{2+c} \times x^2 - \frac{2+c}{2+c} \times bx + b^2 = a^2$: Whence

$$x^2 - bx = \frac{aa - bb}{2 + c}, \text{ and } x = \sqrt{\frac{aa - bb}{2 + c} + \frac{1}{4}bb} + \frac{b}{2}.$$

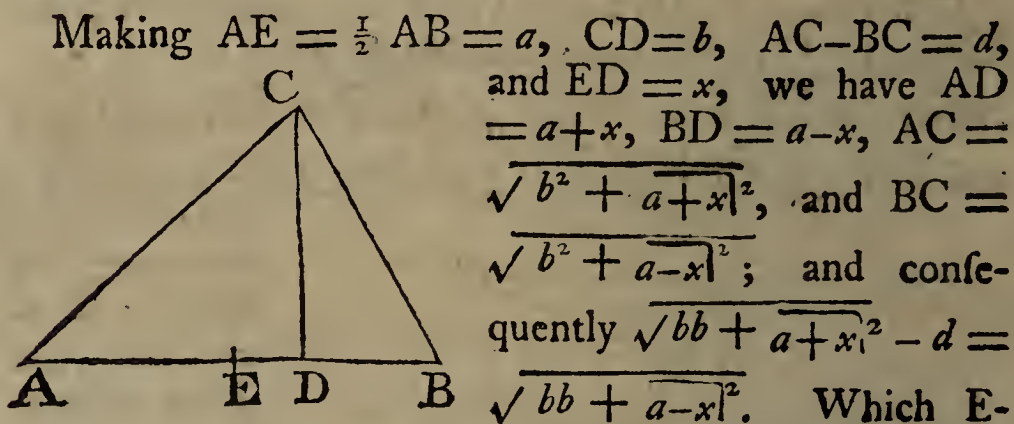
Geometrically.

The Geometrical Construction of this Problem, as the Angle ADH is given ($= \frac{1}{2}$ a Right Angle) is exceeding obvious: For, if DE be supposed $= DC$, so that AE may express the given Difference of AD and CD, the Angle DEC (supposing CE drawn) will be given $= \frac{1}{2}$ ADH. Therefore the Triangle AEC, by Means of the given Angle AEC, and the two given Sides AE and AC, may be constructed. And then, by producing

portional to $AE-BE$, AE , and BE , *it is there demonstrated*, that two Lines drawn from A and B to meet any-where in the Circumference of a Circle described thro' E , from the Center O , will be in the same given Ratio of AE to BE . Whence it is evident that the Intersection of the said Circumference with a Right-line FG , drawn parallel to AB , at the given Distance DC , will determine the Vertex of the Triangle.

P R O B L E M XX.

Having the Base AB , the Perpendicular CD , and the Difference of the two Sides, AC and BC of a Triangle ABC ; to find the Sides.



Making $AE = \frac{1}{2} AB = a$, $CD = b$, $AC - BC = d$, and $ED = x$, we have $AD = a + x$, $BD = a - x$, $AC = \sqrt{b^2 + a + x}^2$, and $BC = \sqrt{b^2 + a - x}^2$; and consequently $\sqrt{bb + a + x}^2 - d = \sqrt{bb + a - x}^2$. Which Equation, squared, gives $bb + a + x^2 - 2d \sqrt{bb + a + x}^2 + dd = bb + a - x^2$; and this, by Reduction, becomes $4ax + dd = 2d \sqrt{bb + a + x}^2$. And this, again squared, produces $16a^2x^2 + 8addx + d^4 = 4dd \times bb + aa + 2ax + xx$; or, $16a^2x^2 + d^4 = 4dd \times aa + bb + 4ddxx$. Whence $x = \sqrt{\frac{4dd \times aa + bb - d^4}{16aa - 4dd}}$.

The Geometrical Construction of this Problem being only a particular Case of a more general One, given at large hereafter (*Problem 49*) I shall not insert it here: But observe, with respect to the Algebraical Solution, that, if d be supposed to denote the Sum, instead of the Difference, of the Sides, the Value of x (or DE) will be

be given by the very Equation above exhibited; as is manifest from the Process.

PROBLEM XXI.

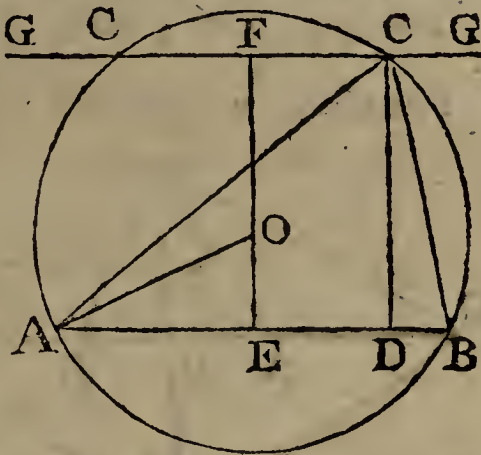
Having the Base AB, the Perpendicular CD, and the Rectangle of the two Sides AC and BC, of a Triangle ABC; to determine the Triangle.

By retaining the Notation of the preceding Problem, and making the given Rectangle = c^2 , we have

$$\sqrt{bb + a + x}^2 \times \sqrt{bb + a - x}^2 = c^2; \text{ and therefore}$$

$$bb + a + x^2 \times bb + a - x^2 = c^4; \text{ that is } b^4 + b^2 \times 2a^2 + 2x^2 + a^4 - 2a^2 x^2 + x^4 = c^4.$$

$$\text{Whence } x^4 + 2bb - 2aa \times x^2 = c^4 - b^4 - 2a^2 b^2 - a^4.$$



$$\text{Which, solved, gives } x = \sqrt{aa - bb \pm \sqrt{c^4 - 4a^2 b^2}}.$$

Geometrically.

The Magnitude of the Rectangle under the two unknown Lines, AC and BC, being given, two other Lines must, therefore, be assigned, containing an equal Rectangle; whereof One being given, the Other will also become known. (*Vid. Observation 5, P. 88.* But it is known that the Rectangle under the, given, Perpendicular CD and the Diameter of a Circle circumscribing the Triangle, is equal to the Rectangle under the said, unknown, Sides of the Triangle (*Elem. 19. 3*); Hence the Diameter of the circumscribing Circle is given; and from thence the following Construction.

Find a Third-Proportional to CD and the Side of the Square, expressing the Magnitude of the proposed Rectangle; and with the Half Thereof, from the Point A (or B) describe an Arch, cutting EF, perpendicular to AB,

AB, in O; from which Point, as a Center, with the same Radius, let a Circle ACB be described; then a Right-line, GFG, drawn parallel to AB, at the given Distance DC, will intersect the said Circle in the Vertex of the Triangle.

P R O B L E M XXII.

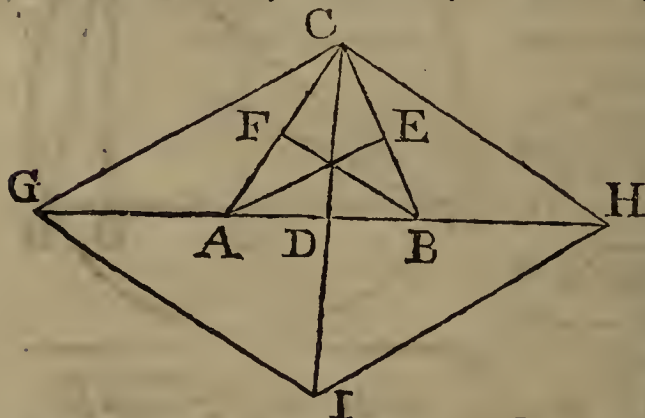
The Lengths of three Lines AE, BF and CD, drawn from the Angles to the Middle of the opposite Sides of a Triangle, being given; to find the Sides.

Put $CD = a$, $BF = b$, $AE = c$, $AB = x$, $AC = y$

and $BC = z$. Since, by a known Property of Triangles,

$$\overline{AC}^2 + \overline{BC}^2 = 2\overline{CD}^2 + 2\overline{AD}^2$$

(Elem. 12. 2) we have $y^2 + z^2 = 2a^2 + \frac{1}{2}x^2$, and there-



fore $y^2 + z^2 - \frac{1}{2}x^2 = 2a^2$. In the same manner

$$x^2 + z^2 - \frac{1}{2}y^2 = 2b^2.$$

$$x^2 + y^2 - \frac{1}{2}z^2 = 2c^2.$$

From whence (by taking the former of these Equations from twice the Sum of the two Latter) there comes out

$$4x^2 + \frac{1}{2}x^2 = 2 \times 2b^2 + 2c^2 - a^2: \text{ And consequently}$$

$$x = \frac{2}{3} \sqrt{2b^2 + 2c^2 - a^2}. \text{ By the same Argument, } y =$$

$$\frac{2}{3} \sqrt{2a^2 + 2c^2 - b^2}; \text{ and } z = \frac{2}{3} \sqrt{2a^2 + 2b^2 - c^2}.$$

Geometrically.

If CG and CH be drawn parallel to AE and BF, meeting AB, produced, in G and H; it is plain, because $CE = BE$, and $CF = AF$, that $AG = AB = BH$; and also that $CG = 2AE$, and $CH = 2BF$. Therefore, the two Sides CG, CH, and the Line CD, bisecting the Base of the Triangle GCH being given, the Diagonal CI ($= 2CD$) of the Parallelogram GCHI (as well

well as the Sides) will be given (*Elem.* 13. 2). Hence, in order to the Construction, let a Triangle CGI, whose three Sides are equal to the Doubles of the three given Lines, be constituted, and draw GDH to bisect CI in D; also set off DA and DB, each equal to $\frac{1}{3}$ of GD; join A, C, and B, C, and the Thing is done.

From this Construction we have the very same Numerical Solution as from the algebraic Process: For, since, $2\overline{GD}^2 + 2\overline{CD}^2 = \overline{CG}^2 + \overline{CH}^2$ (*Elem.* 12. 2.) $= 4\overline{AE}^2 + 4\overline{BF}^2$, thence is $GD =$

$$\sqrt{2\overline{AE}^2 + 2\overline{BF}^2 - \overline{CD}^2}, \text{ and consequently } AB$$

$$(\frac{2}{3} GD) = \frac{2}{3} \sqrt{2\overline{AE}^2 + 2\overline{BF}^2 - \overline{CD}^2}. \text{ By}$$

the Construction it also appears that no one of the three given Lines must be greater than the Sum of the other two.

PROBLEM XXIII.

All the Sides of a Triangle ABC being given; to find the Perpendicular CD, the Segments of the Base AD and BD, together with the Area of the Triangle.

Put $AC=a$, $AB=b$, $BC=c$, and $AD=x$: Then $BD=b-x$; and $c^2 -$

$$\overline{b-x}^2 (= \overline{CD}^2) = a^2$$

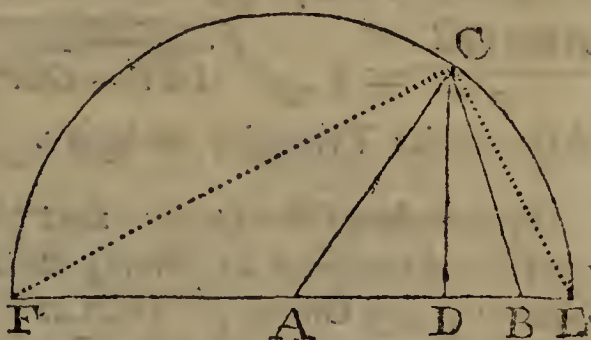
$$- x^2; \text{ that is, } c^2 -$$

$$b^2 + 2bx - x^2 = a^2 -$$

$$x^2. \text{ Whence } 2bx =$$

$$a^2 + b^2 - c^2, \text{ and } x =$$

$$\frac{aa+bb-cc}{2b}.$$



$$\text{Now } \overline{CD}^2 = \overline{AC}^2 - \overline{AD}^2 = \overline{AC+AD} \times \overline{AC-AD}$$

$$= a + \frac{aa+bb-cc}{2b} \times a - \frac{aa+bb-cc}{2b} = \frac{aa+2ab+bb-cc}{2b} \times$$

$$\frac{-aa+2ab-bb+cc}{2b} = \frac{\overline{b+a}^2 - c^2}{2b} \times \frac{c^2 - \overline{b-a}^2}{2b}.$$

I

Hence

Hence $CD = \frac{1}{2b} \sqrt{(b+a)^2 - c^2 \times c^2 - (b-a)^2}$; and
the Area $\left(\frac{CD \times AB}{2}\right) = \frac{1}{4} \sqrt{(b+a)^2 - c^2 \times c^2 - (b-a)^2}$.

Geometrically.

From the Center A, with the Radius AC, let a Semi-circle ECF be described, cutting AB produced in E and F; so that BF may be the Sum, and BE the Difference, of the Sides AC and AB: Also let EC and FC be drawn.

Then will $\overline{FC}^2 = \overline{FE} \times \overline{FD} (= 2AF \times FD)$; and $\overline{EC}^2 = \overline{FE} \times \overline{ED} (= 2AE \times ED)$, by *Elem. Corol. to 12. 4.*

Also $\overline{BC}^2 = \overline{BF}^2 + \overline{FC}^2 - 2BF \times FD (= 2AF \times FD) - 2FB \times FD = \overline{BF}^2 - 2AB \times FD$ (*Elem. 8. 2*): And likewise $\overline{BC}^2 = \overline{BE}^2 + \overline{EC}^2 - 2BE \times ED (= 2AE \times ED) - 2EB \times ED = \overline{BE}^2 + 2AB \times ED$.

Hence it appears that $2AB \times FD = \overline{BF}^2 - \overline{BC}^2$; and $2AB \times ED = \overline{BC}^2 - \overline{BE}^2$: And, consequently, that $\overline{BF}^2 - \overline{BC}^2 \times \overline{BC}^2 - \overline{BE}^2 = 4\overline{AB}^2 \times \overline{FD} \times \overline{ED} = 4\overline{AB}^2 \times \overline{DC}^2$ (by *Elem. Cor. to 12. 4*). Therefore $\frac{AB \times DC}{2} = \frac{1}{4} \sqrt{\overline{BF}^2 - \overline{BC}^2 \times \overline{BC}^2 - \overline{BE}^2}$ = the Area of the Triangle, as before.

From whence it appears, that the Area of any Triangle will be determined by finding the Differences between the Square of any One of its Sides and the Squares of the Sum, and Difference, of the other Two; and then taking $\frac{1}{4}$ of the square Root of the Product arising by the Multiplication of the said Differences into each other.

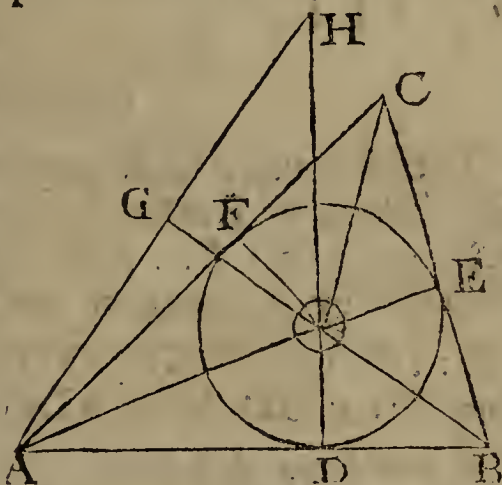
PROBLEM

PROBLEM XXIV.

Having all the Sides of a Triangle ABC; to find the Radius of its inscribed Circle DEF.

From the Center O, to the angular Points and the Points of Contact, let Lines be drawn; and, upon BO produced, let fall a perpendicular AG.

It is plain, in the first Place (because $OD=OE=OF$) that $AD=AF$, $BD=BE$, and $CF=CE$: Therefore, by Addition, $BD+CF (=BE+CE) = BC$: Take each of these equal Quantities from $AB+AC$, and there will remain $AD+AF = AB+AC-BC$; From whence (AD being $= AF$) we get AD (or AF) $= \frac{AB+AC-BC}{2}$. By subtracting of which from AB ,



and from AC , we also have $BD = \frac{AB+BC-AC}{2}$, and $CF = \frac{AC+BC-AB}{2}$.

Moreover, it is evident that the Triangles AOG and COF are similar: For the Sum of all the Angles at the Center, $DOE+DOF+FOE$ being $= 4$ Right Angles, the Sum of their Halves, $BOD+DOA+COF$, must be $= 2$ Right Angles $= BOD+DOA+AOG$; and consequently $COF = AOG$.

Now let the Values of AD , BD , and CF (found above) be denoted by a , b and c , respectively; and put OD ($OE=OF$) $= x$: Then, it will be BO ($\sqrt{bb+xx}$): OD (x) :: AB ($a+b$): $AG = \frac{ax+bx}{\sqrt{bb+xx}}$; and BO :

$BD :: AB : BG = \frac{ab + bb}{\sqrt{bb + xx}}$. Therefore $OG (BG +$

$BO) = \frac{ab + bb}{\sqrt{bb + xx}} - \sqrt{bb + xx} = \frac{ab - xx}{\sqrt{bb + xx}}$. But, $AG :$

$OG :: CF : OF$; Or, $ax + bx : ab - xx :: c : x$
whence $ax^2 + bx^2 = abc - cx^2$; and consequently $x =$

$$\sqrt{\frac{abc}{a + b + c}}.$$

Geometrically.

Let DO and AG be produced to meet each other in H . Then, by Reason of the similar Triangles, it will be, $AB : HO (:: AG : OG) :: CF : FO$; And therefore by Alternation and Composition, $AB + CF : CF :: HO + FO (HD) : FO (OD) :: HD \times OD : \overline{OD}^2$. But $HD \times OD$ is $= AD \times BD$ (*Elem.* 18. 3): Therefore we have $AB + CF : CF :: AD \times BD : \overline{OD}^2 = \frac{AD \times BD \times CF}{AD + BD + CF}$, the very same as before.

From this Conclusion, the Rule in common Practice, for finding the Area of a Triangle, having the three Sides given, is easily deduced: For it is evident that the Area of the Triangle ABC is equal to the Radius (OD) drawn into the Half Sum of the Sides ($AD + BD$

$+ CF$); that is $= \sqrt{AD + BD + CF} \times AD \times BD \times CF$.

Where AD , BD , and CF , are the Differences between the Half Sum and each particular Side.

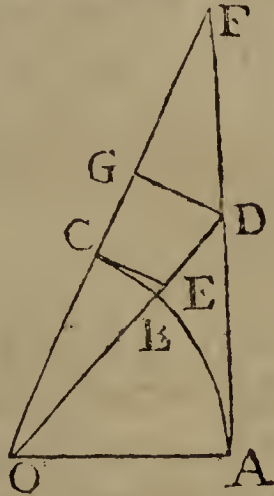
PROBLEM

PROBLEM XXV.

The Radius of a Circle and the Tangents of two Arcs thereof being given, to determine the Tangent of the Sum of those Arcs.

Let AB and BC be the proposed Arcs, whereof the given Tangents are AD and CE; and let the former of These be continued out to meet the Radius OC, produced, in F; so shall AF be the Tangent of AC, the Sum of the said Arcs.

Now, calling AO, r ; AD, m ; CE, n ; AF, x ; and FO, y ; and making DG perpendicular to FO; we have (by sim. Triangles) OF (y) : AO (r) :: DF ($x-m$) : DG = $\frac{rx-rm}{y}$.



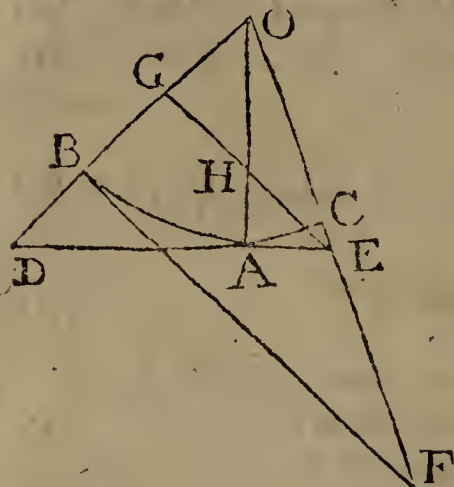
Also OF (y) : AF (x) :: DF ($x-m$) : FG = $\frac{xx-mx}{y}$.

From which last we have OG (= OF - FG) = $y - \frac{xx-mx}{y}$.
 $\frac{xx-mx}{y} = \frac{yy-xx+mx}{y} = \frac{rr+mx}{y}$ (because $yy-xx=rr$).

But OG ($\frac{rr+mx}{y}$) : DG ($\frac{rx-rm}{y}$) :: OC (r) : CE (n) and consequently $r^2x-r^2m=r^2n+mnx$. Whence
 $x = \frac{rr \times m + n}{rr - mn}$; or AF = $\frac{AO^2 \times AD + CE}{AO^2 - AD \times CE}$.

Otherwise.

Let AD and AE be the Tangents of the two Arcs AB and AC, and BF That of their Sum BC; also let EG be drawn perpendicular to OD, intersecting the Radius OA in H. Then, by Reason of the similar Triangles, it will be $AO : AD :: AE : AH$. And, $OH (AO - AH) : DE$



$$(\therefore : OG : GE) :: OB (AO) : BF = \frac{AO \times DE}{AO - AH} =$$

$$\frac{AC^2 \times DE}{AO^2 - AO \times AH} = \frac{AO^2 \times AD + AE}{AO^2 - AD \times AE}; \text{ because,}$$

by the first Proportion $AO \times AH = AD \times AE$. Which Conclusion is the very same with That above.

If the Tangents of the Arcs AC and AB (*Fig. 1.*) were to be given, in order to find the Tangent of their Difference BC; then, by the Proportion $\frac{rr + mx}{y} :$

$$\frac{rx - rm}{y} :: r : n \text{ (above derived) we should have } n =$$

$$\frac{rr \times x - m}{rr + mx}; \text{ or } CE = \frac{AC^2 \times AF - AD}{AO^2 + AF \times AD}.$$

PROBLEM XXVI.

The Ratio of the Sines DE, FG of two Arcs AD, AF, of a given Circle, together with That of their Tangents AB, AC being given; to find both the Sines and the Tangents.

Put the Radius $AO = a$; and let the given Ratio of AB to AC be That of m to n ; moreover let DE be to FG

FG as p to q ; and call AB, x ; and AC, y : Then
(by similar Triangles) $\overline{OB}^2 (a^2 + x^2) : \overline{AB}^2 (x^2) ::$

$$\overline{OD}^2 (a^2) : \overline{DE}^2 = \frac{a^2 x^2}{aa + xx};$$

$$\text{and } \overline{OC}^2 (a^2 + y^2) : \overline{AC}^2 (y^2)$$

$$:: \overline{OF}^2 (a^2) : \overline{FG}^2 = \frac{a^2 y^2}{aa + yy}.$$

Therefore, by the Question, $p^2 :$

$$q^2 :: \frac{a^2 x^2}{aa + xx} : \frac{a^2 y^2}{aa + yy}, \text{ and}$$

consequently $p^2 y^2 \times a^2 + x^2 = q^2 x^2 \times a^2 + y^2$. But, by

the Question, we also have, $m : n :: x : y, = \frac{nx}{m}$;

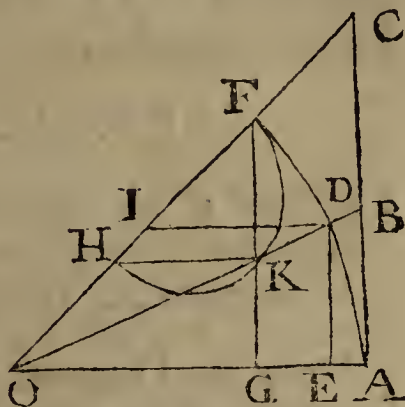
which Value, substituted in the preceding Equation,

$$\text{gives } \frac{n^2 p^2 x^2}{mm} \times a^2 + x^2 = q^2 x^2 \times a^2 + \frac{nnxx}{mm} : \text{Whence}$$

$$n^2 p^2 \times a^2 + x^2 = q^2 \times a^2 m^2 + n^2 x^2; \text{ and } x = \frac{a}{n} \times$$

$$\sqrt{\frac{nnpp - mmqq}{qq - pp}}. \text{ From which AC, DE and FG}$$

are also given.



Geometrically.

Since the Ratio of AB to AC is given, as m to n ; GK (supposing K to be the Intersection of FG and OB) will be to GF, in the same given Ratio: And, if (agreeable to the 3^d General Observation) KH be drawn parallel to AO, meeting OF in H, OH will be to OF still in the same given Ratio.

Again, if DI be drawn parallel to AO, meeting OF in I; then $OK : OH :: OD (OF) : OI :: FG : DE :: q : p$; Whence OK is also given; and from thence the following Construction.

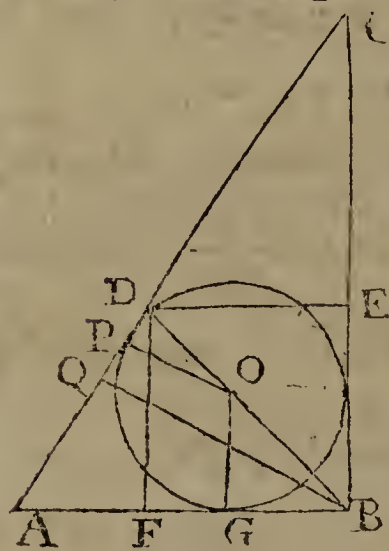
In any Radius OF of the given Circle, take OH to OF in the given Ratio of m to n ; and upon HF let a Semi-circle be described: Take also a Fourth proportional

portional to p , q , and OH ; with which, as a Radius, from the Center O , describe an Arch, cutting the Semi-circle in K ; and, having drawn HK , make OA parallel thereto.

PROBLEM XXVII.

The Side of the Square, and the Radius of the Circle, inscribed in a right angled Triangle ABC , being given; to determine the Triangle.

Draw the Diagonal BD of the Square; and from the Center O of the given Circle, to the Points of contact,



draw the Radii OG and OP ; and upon the Hypotenuse AC let fall the Perpendicular BQ : Calling the Side of the Square, a ; the Radius of the Circle, b ; and AQ , x : Then, because of the parallel Lines, we shall have $FG(a-b) : BF(a) :: (OD : BD ::) GP(b) : BQ = \frac{ab}{a-b}$:

Whence $DQ (= \sqrt{BD^2 - BQ^2})$
 $= \sqrt{2a^2 - \frac{aabb}{a-b}^2}$, is also given.

Let it, for Brevity-sake, be denoted by c ; and let $EQ \left(\frac{ab}{a-b} \right) = d$: Then we shall have, $AQ(x) : BQ(d) :: BQ(d) : CQ = \frac{dd}{x}$; and also (by *Elem. 10. 4.*) $AD(x+c) : CD \left(\frac{dd}{x} - c \right) :: AB : BC :: AQ(x) : BQ(d)$ whence, by multiplying Extremes and Means, we get $dx + cd = dd - cx$; and, from thence, $x = \frac{d \times d - c}{d + c}$. By Means whereof every thing else is readily found.

Geometrically.

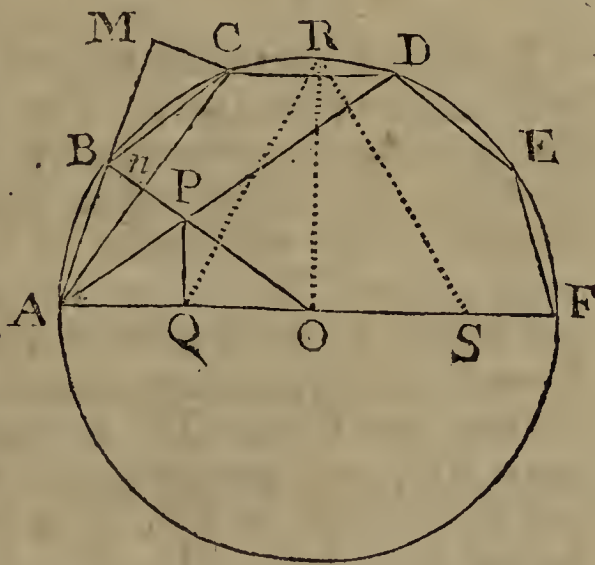
Geometrically.

The Geometrical Construction, and the Trigonometrical Solution of this Problem are very easy. For, since the Position of the Point D with respect to the Circle, is given; a Right-line DPA drawn from thence (*by Elem. 16. 5.*) to touch the Circle, will determine the Triangle: And then, from the given Lines OP and OD, the Angle D may be found: By Means whereof and ABD ($= \frac{1}{2}$ a Right Angle) together with the given Side BD, all the Rest will become known.

PROBLEM XXVIII.

*To determine the Sides of a regular Pentagon and Decagon,
inscribed in a given Circle.*

Let AB, BC, CD, &c. be Sides of the Decagon, and AC a Side of the Pentagon: And let AD be drawn, intersecting the Radius OB in P. It is evident in the first Place, that the Angles BAP and OAP, standing on the equal Arches BD and BF, are equal to one another, and also equal, each of them, to the Angle AOP, insisting on the Arch AB (*Elem.* 9. 3.) And, secondly, that the Triangle BAP (as well as APO) is an isosceles One, because the Perpendicular AnC, makes equal Angles BAC, DAC with the two Sides AB, AP of the Triangle. Hence it appears very plain that all the three Lines AB, AP, and OP are equal among themselves; and likewise that $AO (OB) : AB (OP) :: OP (AB) : BP$ (*by Elem.* 10. 4.) seeing the Angle BAO



is bisected by AP. Moreover, by letting fall the two Perpendiculars PQ and CM, upon AO and ABM, the Triangles BCM and APQ (as BC is = AB=AP, and the Angle MBC = MAD = PAQ) will appear to be equal in all respects; and so, BM being (=AQ) = $\frac{1}{2}$ AO, we have $\overline{AC}^2 (= \overline{BC}^2 + \overline{AB}^2 + 2BM \times AB, \text{Elem. II. 2.}) = \overline{BC}^2 + \overline{AB}^2 + AO \times AB$. But, by the above Proportion, \overline{AB}^2 is = $AO \times BP$: Therefore, by writing $AO \times BP$ in the Room of \overline{AB}^2 , we get $\overline{AC}^2 = \overline{BC}^2 + AO \times AB + AO \times BP = \overline{BC}^2 + \overline{AO}^2$: Whence (BC being first found) AC will also become known.—If AO be now denoted by a , and AB by x ; then from the Equality of \overline{AB}^2 and $AO \times BP$, you will have $x^2 = a \times a - x$; from which x will be found = $\sqrt{\frac{5aa}{4} - \frac{a}{2}}$; And from thence $AC (= \sqrt{xx + aa}) = a\sqrt{\frac{5 - \sqrt{5}}{2}}$.

As to the Geometrical Construction, it likewise very easily follows from above. For, since the Side of the Decagon appears to be equal to the greater Part of the Radius divided according to Extreme-and-Mean Proportion; and the Square of the Side of the Pentagon exceeds That of the Decagon by the Square of the Radius; the following Solution (given by Dr. Barrow, as an Improvement upon *Euclid's*) is manifest.

Draw the Radius OR, at right-Angles to the Diameter AF; and, having bisected the Radius AO in Q, let QS be taken, in AF, equal to the Distance QR: So shall OF : OS :: OS : FS (*Elem. 22. 5.*) and consequently OS = AB the Side of the Decagon.

And, because \overline{KS}^2 (supposing RS drawn) is = $\overline{OS}^2 + \overline{OR}^2$, it is plain also that RS will be equal to the Side AC of the Pentagon.

PROBLEM

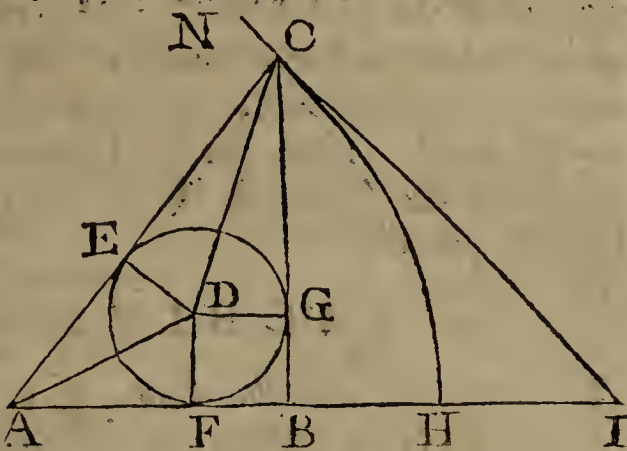
P R O B L E M XXIX.

Having the Hypotenuse AC of a right-angled Triangle ABC, and also the Radius of the inscribed Circle DEFG; to find the two Legs AB and BC.

From the Center D of the given Circle, to the Points of Contact, let DE, DF, and DG be drawn; also draw AD and CD:

And put $DE (= DG = DF) = a$,
 $AC = b$, $AB = x$,
 and $BC = y$.

It is evident, that CE must be $= CG = y - a$; because the right-angled Triangles CDE and CDG, having $DE = DG$, and CD common, are equal in all respects. In the very same manner is $AE = AF = x - a$.



Therefore $y - a + x - a = b (= AC)$; from which Equation we have $x + y = b + 2a$. But, from the Property of right-angled Triangles, we also have $xx + yy = bb$. And, if from the Double of This, the Square of the former Equation be subtracted, there will remain $xx - 2xy + yy = bb - 4ab - 4aa$.

From whence, by extracting the square Root, on both Sides, we get $x - y = \sqrt{bb - 4ab - 4aa}$. Which last Equation, added to, and subtracted from, the First, gives $2x = 2a + b + \sqrt{bb - 4ab - 4aa}$, and $2y = 2a + b - \sqrt{bb - 4ab - 4aa}$.

Geometrically.

Seeing the Difference between each Leg of the Triangle and the adjacent Segment of the Hypotenuse, is equal to the Radius of the Circle, it is plain that the Sum of the two Legs ($AB + BC$) will exceed the Sum

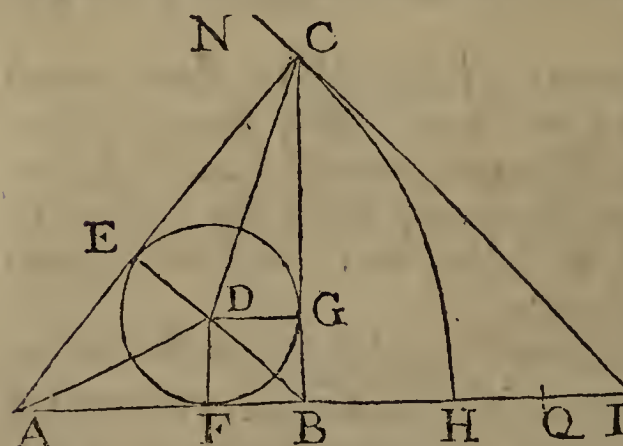
of the two Segments (or the whole Hypothenufe) by twice the Radius ; and will therefore be given $= AC + 2BF$. Whence, if BI be supposed equal to BC, AI will likewise be given $= AC + 2BF$, and the Angle BIC $=$ half a Right-One. Hence the following Construction.

Draw an indefinite Line in which take AH equal to the given Hypothenufe, and HI equal to the Diameter of the given Circle ; also make the Angle AIN equal to half a Right Angle ; and from the Center A, with the Interval AH, let an Arch be described, meeting IN in C ; from which Point upon AP let fall the Perpendicular CB ; so shall AB and BC be the two Legs required.

PROBLEM XXX.

The Perimeter, and the Area, of a right-angled Triangle ABC being given ; to determine the Triangle.

Put the given Perimeter $(AB + BC + AC) = p$, the Area $(\frac{1}{2} AB \times BC) = a^2$; and let half



the Sum of the Legs AB and BC be denoted by x and half their Difference by y : Then, AB being $= x + y$, $BC = x - y$, and $AC = p - 2x$, we shall have

$\overline{x+y} \times \overline{x-y} (= AB \times BC) = 2a^2$; and $\overline{x+y}^2 + \overline{x-y}^2 (= AC^2) = \overline{p-2x}^2$; that is, by Reduction, $xx - yy = 2a^2$; and $2xx + 2yy = pp - 4px + 4xx$.

Now, by the Addition of the latter of these Equations to the Double of the former, there arises $4xx = pp - 4px + 4xx + 4a^2$: Whence x comes out $= \frac{pp + 4aa}{4p} = \frac{1}{4} p + \frac{aa}{p}$. From which Value that of y

$= \sqrt{x^2 - 2a^2}$, will likewise be given.

Geometrically,

Geometrically.

If from the Center D, of a Circle inscribed in the Triangle, Lines be supposed drawn to the angular Points, the proposed Triangle will, by that Means, be divided into three Others ADB, BDC, and ADC; whose Bases are the three Sides of the first Triangle, and their Perpendiculars all Radii of the said Circle. From which it is evident that the Triangle ABC is equal to a Rectangle under Half the Sum of its three Sides (which we will here express by the given Line AQ) and the Radius DF of the inscribed Circle; and consequently that the Radius DF will be given by taking a Third Proportional to AQ and the Side (*a*) of the Square expressing the given Area. Whence, making QH and QI, each, equal to DF, so found; it will appear from the preceding Problem that AH will be = the Hypothenuse, and AI = to the Sum of the two Legs, of the proposed Triangle: Which Quantities being both given, the Method of Construction is manifest *from the last Problem.*

PROBLEM

PROBLEM XXXI.

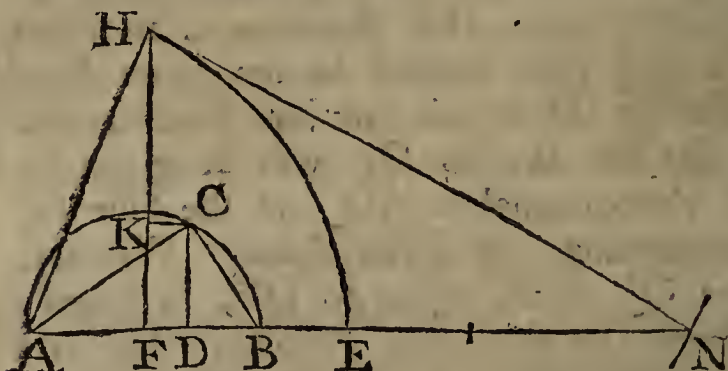
The Sum, or Difference of the two Legs AC and BC of a right-angled Triangle ABC being given, together with the Sum, or Difference of the Hypotenuse AB and a Perpendicular CD falling thereon from the Right-Angle; to find all the Sides of the Triangle.

Put $AC+BC=s$, $AC-BC=d$, $AB+CD=p$,
 $AB-CD=q$:

Then will AC
 $=\frac{s+d}{2}$, $BC=$

$\frac{s-d}{2}$, $AB=$

$\frac{p+q}{2}$, $CD=$



$\frac{p-q}{2}$: But $AB^2 = AC^2 + BC^2$; and $AB \times CD (=$
 $2 \text{ Area } ABC) = AC \times BC$: Which, in Species, give
 $p^2 + 2pq + q^2 = 2s^2 + 2d^2$,
 and $p^2 - q^2 = s^2 - d^2$.

By adding, and subtracting the Double of the last of these Equations from the former, we have these two other Equations,

$$\text{viz. } 3p^2 + 2pq - q^2 = 4s^2;$$

$$\text{and, } -p^2 + 2pq + 3q^2 = 4d^2.$$

From which, when any two of the Quantities, s , d , p , q , are given, the other two will, easily, be determined.

Thus, let s and p be given; then, from the former Equation, we have $q^2 - 2pq = 3p^2 - 4s^2$; whence $q^2 - 2pq + p^2 = 4p^2 - 4s^2$, and $p - q = 2\sqrt{p^2 - s^2}$: Therefore CD
 $(=\frac{p-q}{2}) = \sqrt{p^2 - s^2}$, and $AB = p - \sqrt{p^2 - s^2}$.

If d and q be given; we shall have $CD = \sqrt{q^2 - d^2}$; because $p^2 - q^2 = s^2 - d^2$, or $p^2 - s^2 = q^2 - d^2$ (p. above.)
 But,

But, if s and q be given, then will $3p^2 + 2pq = 4s^2 + q^2$; which, solv'd, gives $p = \frac{2}{3} \sqrt{3ss + qq} - \frac{1}{3} q$.

Lastly, if d and p be given, we shall have $3q^2 + 2pq = 4d^2 + p^2$, and consequently $q = \frac{2}{3} \sqrt{3dd + pp} - \frac{1}{3} p$.

Geometrically.

The very same Properties whereby the algebraical Solution is above brought out, lead us also to Geometrical Constructions of the several Cases of the Problem under Consideration. But it will be sufficient, here, to exhibit That of the Case, wherein the Sum of the Legs (AE) and the Difference of the Hypotenuse and Perpendicular (AF) are supposed given, being the most difficult.

Thus, because $\overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2$,
and $2AC \times BC (= 2AB \times CD) = 2AB \times BF$;
we shall, by adding these equal Quantities,
have $\overline{AC}^2 + \overline{BC}^2 + 2AC \times BC$ (or \overline{AE}^2 , *Elem. 5. 2.*)
 $= \overline{AB}^2 + 2AB \times BF = \overline{AB}^2 + 2AB \times AB - AF =$
 $3\overline{AB}^2 - 2AB \times AF$.

Therefore, if an Arch, from the Center A, with the Radius AE, be described; and, from its Intersection (H) with FH perpendicular to AE, another Arch be also described, with the Radius 2AE, cutting AE produced in N; then the Hypotenuse AB of the required Triangle, will be $\frac{1}{3}$ of the Line AN thus determined.

For \overline{HN}^2 ($4\overline{AE}^2$) being $= \overline{AN}^2 + \overline{AH}^2$ (\overline{AE}^2)
 $= 2AN \times AF$ (*Elem. 9. 2.*), and therefore $3\overline{AE}^2 = \overline{AN}^2$
 $= 2AN \times AF$; it is plain, if AB be taken $= \frac{1}{3} AN$,
that $3\overline{AE}^2 = 3AB \times 3AB - 6AB \times AF$: And consequently that $\overline{AE}^2 = 3\overline{AB}^2 - 2AB \times AF$, *the very same as above.*

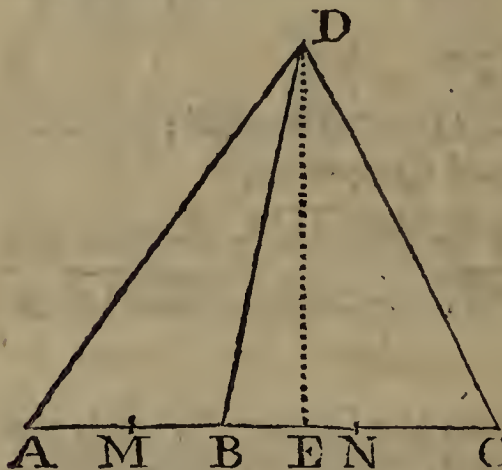
From the Value of AB, thus given, what yet remains to be done, will be effected with great Facility. For, if in FH there be taken $FK = FB$, and KC be drawn

drawn parallel to AN, intersecting a Semicircle, described upon AE, in the Point C, that Point will, it is evident, be the Vertex of the Triangle.

L E M M A.

If a Line be drawn from the Vertex to any Point in the Base of a Triangle, the Sum of the two Solids under the Squares of the two Sides and the alternate Segments of the Base, will be equal to the Solid under the whole Base and its two Segments, together with the Solid under the same Base and the Square of the dividing Line.

Let ACD be any proposed Triangle, and BD the dividing Line; then, I



say, that $\overline{AD}^2 \times BC + \overline{CD}^2 \times AB = AC \times AB \times BC + AC \times \overline{BD}^2$.

For, if AB and CB be bisected in M and N, and a Perpendicular DE be let fall upon AC, it is known

$$\text{that } \left\{ \begin{array}{l} \overline{AD}^2 - \overline{BD}^2 = AB \times 2ME \\ \overline{CD}^2 - \overline{BD}^2 = BC \times 2NE \end{array} \right\} \begin{array}{l} \text{Elem. 8. and} \\ \text{10. of 2.} \end{array}$$

Whence it follows,

$$\text{that } \overline{AD}^2 \times BC - \overline{BD}^2 \times BC = AB \times BC \times 2ME;$$

$$\text{and } \overline{CD}^2 \times AB - \overline{BD}^2 \times AB = AB \times BC \times 2NE.$$

Let these equal Quantities be added together, and the Sums will also be equal;

$$\text{that is, } \overline{AD}^2 \times BC + \overline{CD}^2 \times AB - AC \times \overline{BD}^2 (= AB \times BC \times 2MN) = AB \times BC \times AC; \text{ and consequently } \overline{AD}^2 \times BC + \overline{CD}^2 \times AB = AC \times AB \times BC + AC \times \overline{BD}^2. \quad \text{Q. E. D.}$$

COROL. I. Hence, if $AB = BC$, then will $\overline{AD}^2 + \overline{CD}^2 = 2AB^2 + 2\overline{BD}^2$.

COROL.

COROL. 2. But, if $AD = DC$, then we shall have $\overline{AD}^2 \times BC + \overline{CD}^2 \times AB = \overline{AD}^2 \times BC + \overline{AD}^2 \times AB = \overline{AD}^2 \times AC$: Whence $\overline{AD}^2 \times AC = AB \times BC \times AC + \overline{BD}^2 \times AC$; and consequently $\overline{AD}^2 = AB \times BC + \overline{BD}^2$.

COROL. 3. Lastly, if the Angle $ADB =$ the Angle CDB (or $AD : CD :: AB : BC$, *Elem.* 10. 4.) then $AD \times BC$ being $= CD \times AB$, it follows that $\overline{AD}^2 \times BC = AD \times CD \times AB$; and that $\overline{CD}^2 \times AB = AD \times CD \times BC$. Hence $\overline{AD}^2 \times BC + \overline{CD}^2 \times AB = AD \times CD \times \overline{AB + BC} = AD \times CD \times AC = AB \times BC \times AC + \overline{BD}^2 \times AC$ (*p. above*); and consequently $AD \times CD = AB \times BC + \overline{BD}^2$.

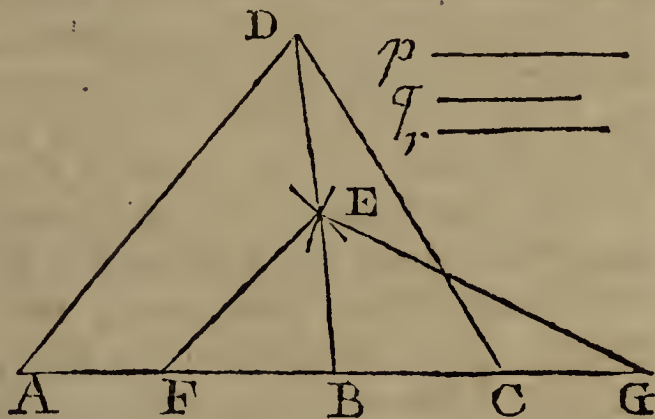
P R O B L E M XXXII.

From three given Points, A, B, C , in the same Right-line, to draw as many Lines, to meet in a fourth Point D , so as to obtain a given Ratio among themselves.

Call AB, a ; BC, b ; AC, c ; and AD, x ; and let AD, BD , and CD be, in Proportion to one another, as p, q , and r , respectively. Then, BD

being $= \frac{qx}{p}$, and

$CD = \frac{rx}{p}$, we



shall, by the preceding Lemma, have $x^2 \times b + \frac{r^2 x^2}{pp} \times a =$

$abc + c \times \frac{q^2 x^2}{pp}$.

Whence $\overline{bp^2 + ar^2 - cq^2} \times x^2 = abc p^2$;
K

And

And consequently $x = p \sqrt{\frac{abc}{b^2p^2 + ar^2 - c^2q^2}}$.

After the same manner the Problem may be resolved, when, instead of the Ratio, the Sums, the Differences, or the Rectangles of the three Lines, are given.

Geometrically.

If from any Point E, in BD, two Lines EF and EG be supposed drawn, so as to form the Angles BEF and BEG, respectively, equal to BAD and BCD, it is evident, from the Similarity of the Triangles BEF, BDA, and BEG, BDC,

that $\begin{cases} BE : BF :: BA : BD \\ BE : BG :: BC : BD. \end{cases}$

And, consequently, that $BF : BG :: BC : BA$. Therefore, if BF be taken $= BC$, BG will be $= AB$.

Moreover, from the abovementioned, similar, Triangles,

we have $\begin{cases} BD : AD (:: q : p) :: BF (BC) : FE \\ BD : CD (:: q : r) :: BG (AB) : GE. \end{cases}$

Whence FE and GE are given; and from thence the following Construction.

Take $BF = BC$ and $BG = AB$; also take a Fourth Proportional to $q : p$, and BC , and Another to q, r , and AB ; And, with These as Radii, from the Centers F and G , let two Arcs be described; and, to their Intersection E , draw BE and FE ; also draw AD , making the Angle $BAD = BEF$, so shall its Intersection with BE , produced, be the Point of Concourse required.

The Trigonometrical Calculation, from this Construction, is very short and easy: For, all the Sides of the Triangle FGE being given, the Angle F may be found; then, in the Triangle BFE , two Sides and the included Angle being known, every Thing else is readily determined.—It may be observed that there is another Construction of this Problem; by Means of the Intersection of two Circles, so described that Lines drawn from the given Points to meet in the Peripheries Thereof, may obtain the given Ratios (*See Elem. 15. 4.*) but the Method given above I look upon as preferable.—As

to

to the Limitations, it is plain the Problem becomes impossible when the two Circles, described from C and F, do not meet each other; that is, when q is given either, less than the Difference, or greater than the Sum of $\frac{pb}{c}$ and $\frac{ra}{c}$.

PROBLEM XXXIII.

The two Sides AD, CD, of a Triangle being given in Length, together with the Length of a Line DB dividing the Base AC in a given Ratio; to determine the Base, or Line so divided.

Call AD, a ; BD, b ; CD, c ; and AB, x ; and let the given Ratio of AB to BC be that of m to n . Hence

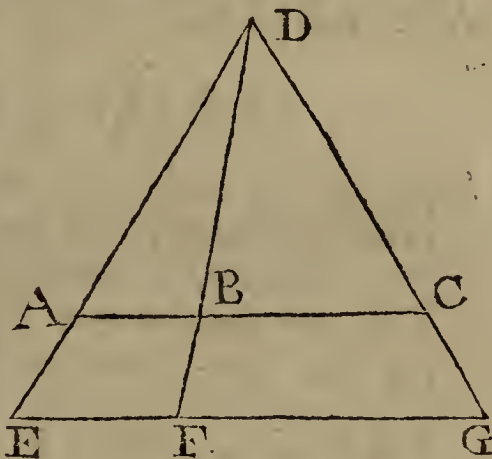
$$BC = \frac{nx}{m}, \text{ and } AC (= x$$

$$+ \frac{nx}{m}) = \frac{m+n \times x}{m}. \text{ And}$$

therefore, by the Lemma,

$$a^2 \times \frac{nx}{m} + c^2 \times x = \frac{m+n \times x}{m}$$

$$\times \frac{nx}{m} \times x + \frac{m+n \times x}{m} \times b^2.$$



Whence, by Reduction, $mna^2 + m^2c^2 - m \times m + n \times b^2 = m + n \times nx^2$.

$$\text{And } x = \sqrt{\frac{mna^2 + mmcc}{m + n \times n} - \frac{mbb}{n}}.$$

If DB be supposed to bisect the Base; then, m and n

$$\text{being equal, } x \text{ becomes } = \sqrt{\frac{aa + cc}{2} - bb}.$$

But, if DB be supposed to bisect the vertical Angle, we shall have $m : n (:: AD : CD) :: a : c$ (*Elem.* 10. 4).

K 2

Whence,

Whence, by writing a and c instead of m and n , our

Equation becomes $x (= \sqrt{\frac{a^3c + a^2c^2}{a+c \times c} - \frac{ab^2}{c}}) = \sqrt{a^2 - \frac{ab^2}{c}}.$

Geometrically.

In any Right-line EG, at pleasure, let there be taken EF and FG in the given Ratio of AB to BC; and, from the Points E, F and G, let three Lines be drawn to meet in a Point D (*by the last Problem*) so as to obtain the given Ratio of AD, BD, and CD, respectively: Then, if DA be taken of the given Length, and ABC be drawn parallel to EFG, it is manifest that ADC will be the Triangle required.

P R O B L E M XXXIV.

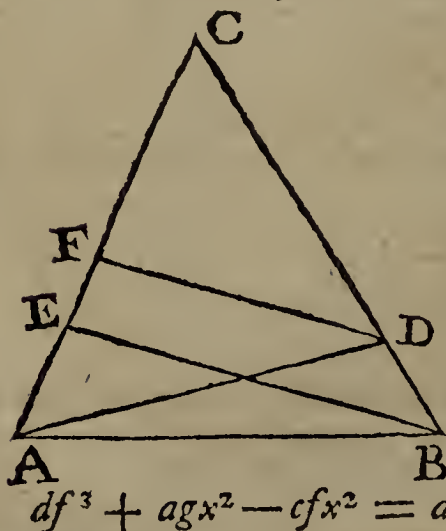
Supposing two Sides AC, BC, of a Triangle ABC, to be given in Length; and that two Lines BE, AD, drawn from the opposite Angles, to cut off given Segments AE, BD, are equal to each other; 'tis proposed to determine the other Side AB of the Triangle.

Put $CE = a$, $AE = b$, $CD = c$, $BD = d$, $CA = f$, $CB = g$, $AB = x$, and AD (or BE) $= z$. Then, *by the Lemma*, we have the two following Equations.

$$g^2 \times b + x^2 \times a = fab + f \times z^2,$$

$$f^2 \times d + x^2 \times c = gcd + g \times z^2.$$

From the former of which, multiply'd by g , let the latter, multiply'd by f , be subtracted, and there will arise $bg^3 -$



$$df^3 + agx^2 - cf x^2 = abfg - cd fg.$$

Therefore $x = \sqrt{\frac{ab - cd \times fg + df^3 - bg^3}{ag - cf}}.$

Geometrically.

Geometrically.

If DF be drawn parallel to BE, the Ratio Thereof to BE (or AD) will be given, as CD to CB; and CF will likewise be to CE in the same given Ratio, and therefore will be given in Length. Hence it is evident that the Position of the Point D with respect to the Side AC (first laid down) will be determined by the Intersection of two Circles; One described from the Center C, with the given Interval CD; and the Other, by *Elem.* 15. 4, so that Lines drawn from F and A, to meet any-where in the Periphery Thereof, may obtain the said given Ratio of CD to CB. Through which Point, so determined, the other Side CB of the Triangle must be drawn; and then, AB being joined, the Thing is done.

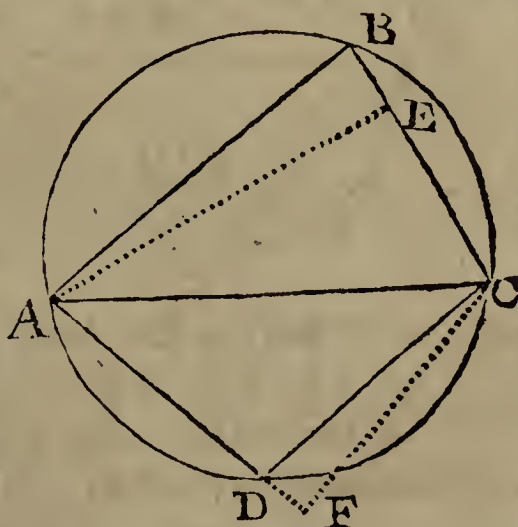
P R O B L E M XXXV.

All the Sides of a Trapezium ABCD, about which a Circle may be inscribed, being given in Length; to determine the Diameter of the Circle.

Let the Diagonal AC be drawn, and upon BC and AD let fall the Perpendiculars AE and CF. And put $AB=a$, $BC=b$, $CD=c$, $AD=d$, and $BE=x$.

Now the external Angle CDF being equal to the internal, opposite, Angle B (*Elem. Corol. to 13. 3.*) the Triangles ABE and CDF are similar: And therefore $AB(a) : BE(x)$

$$:: CD(c) : DF = \frac{cx}{a}. \text{ But } \overline{AB}^2 + \overline{BC}^2 - 2BC \times BE$$



$(= \overline{AC})^2 = \overline{AD}^2 + \overline{CD}^2 + 2AD \times DF$; that is, in Species, $aa + bb - 2bx = cc + dd + \frac{2cdx}{a}$.

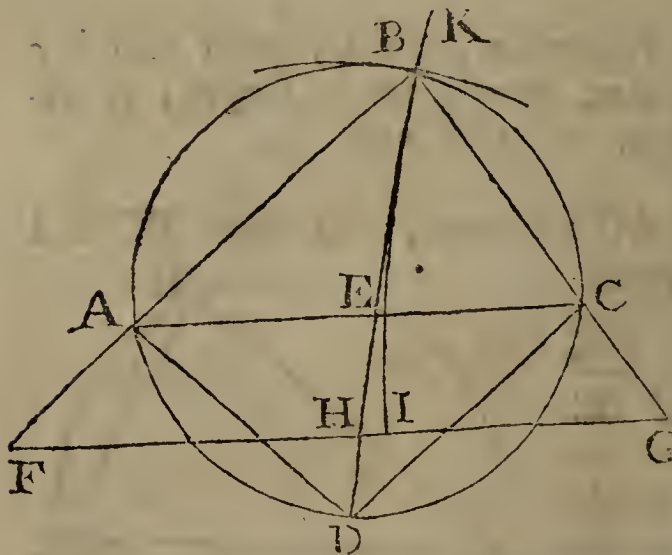
From which Equation x is found $= \frac{aa + bb - cc - dd}{2b + \frac{2cd}{a}}$.

Whence $AE (= \sqrt{aa - xx})$ and $AC (= \sqrt{aa + bb - 2bx})$ will also be given: And then it will be $AE : AC :: AB$: the Diameter sought (*Elem.* 19. 3).

Geometrically.

If the two Diagonals of the Trapezium be drawn, intersecting each other in E , the Triangles ABE and CED , as well as CEB and AED , will be equiangular (*Elem. Corol. to 9. 3*).

Whence $\begin{cases} AB : CD :: BE : CE \\ BC : AD :: BE : AE. \end{cases}$



And, since the Ratios of CE and AE to BE are thus given, it follows, that, if any Line FG be drawn parallel to AC , the Parts thereof GH and FH , intercepted by BC , BE , and BA (produced)

will be to BH in the same given Ratios,

Or that $\begin{cases} AB : CD :: BH : GH \\ BC : AD :: BH : FH; \end{cases}$

and consequently, that, if BH be taken $= AB$, GH will be given $= CD$, and $FH = \frac{AD \times AB}{BC}$. Whence

the following Construction.

Make

Make FH a Fourth-Proportional to BC, AB, and AD (*by Elem. 13. 5*); and, in the same Line produced, take $HG = CD$: Then (*by Elem. 15. 4.*) let a Circle IK be described, so that two Lines drawn from F and G, to meet any-where in the Periphery Thereof, may obtain the given Ratio of AB to CB. And from H, as a Center, with the Radius AB, let another Arch be described, intersecting the Former in B; draw BF and BG, in which set off BA and BC of the given Lengths; then, thro' the three Points A, B, and C, let a Circle be described, and the Thing is done.

P R O B L E M XXXVI.

All the Sides and, the Area of a Trapezium ABCD being given; to determine the Trapezium.

Suppose the Diagonal AC to be drawn; suppose also CE and CF to be perpendicular to AD and AB: And make $AD = a$, $DC = b$, $BC = c$, $AB = d$, the given Area $ABCD = r^2$, $DE = x$, and $BF = y$.

Then will $aa + bb + 2ax (=AC^2) = cc + dd + 2dy$ (*Elem. 11. 2.*) and $a\sqrt{bb-xx} + d\sqrt{cc-yy} (=2ADC + 2ABC) = 2r^2$ (*by the Question*).

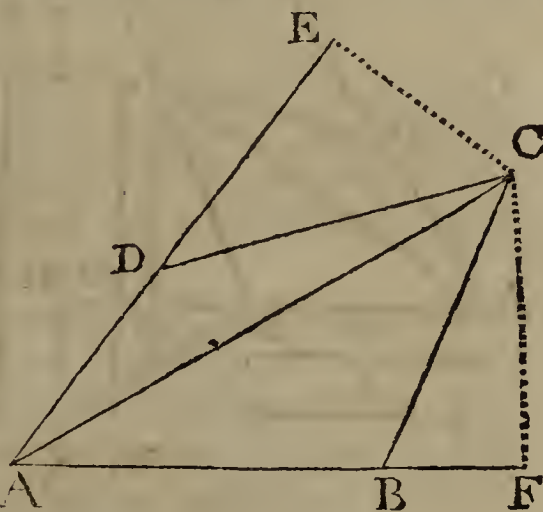
Put $c^2 + d^2 - a^2 - b^2 = 2f$; and, by the first Equation, you will have $ax - dy = f$.

Moreover, by squaring the two last Equations, and then adding them together, you will have

$$a^2b^2 + c^2d^2 + 2ad\sqrt{bb-xx} \times \sqrt{cc-yy} - 2adxy = 4r^2 + f^2.$$

Which, by dividing by $2ad$, and making $g = \frac{4r^2 + f^2}{2ad} - \frac{ab^2}{2d} - \frac{c^2d}{2a}$, is reduced to $\sqrt{bb-xx} \times \sqrt{cc-yy} = g + xy$.

This, squared, gives $b^2c^2 - b^2y^2 - c^2x^2 = g^2 + 2gxy$:
K 4 Which,



Hence we have $DE - BQ = \frac{L^2}{2AD}$, and $AD \times EC + AB \times CF (= 2r^2) = AD \times EC + AD \times PQ$: Whence, by taking a Third Proportional (T) to $2AD$, and L , and Another (V) to $\frac{1}{2}AD$ and r , there results $DE - BQ = T$; and $EC + PQ = V$.

Therefore the Problem is reduced to this; to find two Angles CDE , PBQ , so that the Sum of their Sines CE , PQ , and the Difference of their Co-sines DE , BQ (answering to given, but unequal, Radii DC , BP) may be both given Quantities.

And the Construction thereof (which is exceeding evident) will be as follows.

Draw two Lines cutting each other at Right-Angles in M , in which take $MP = V$, the given Sum of the Sines; and $MN = T$, the given Difference of the Co-sines: Then, from the Centers P and N , with the given Radii $PB (= \frac{AB \times BC}{AD})$ and DC , let two Arcs be described, intersecting each other in B ; through which Point draw OA parallel to MN ; and join B , P , and B , N ; so shall PBO and $NBO (= CDE)$ be the two Angles required. Which being known, the Trapezium itself is very easily constructed.

This Problem, it may be observed, becomes impossible when the two Arcs, described from the Centers P and N , do not meet, but fall short of each other; that is, when the given Area is greater than That of a Trapezium of the same given Sides, inscribed in a Circle, determined by the preceding Problem.

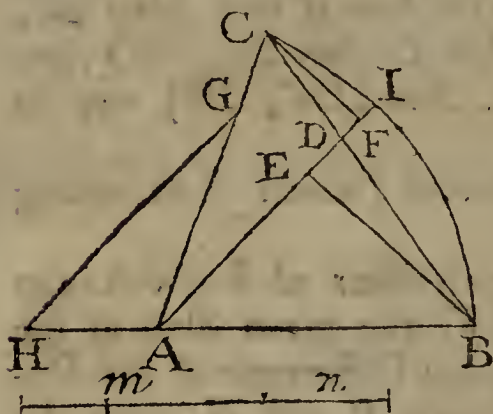
But, besides the above, there is another Limit, for the least Value of the Area (except in one particular Case) and the Problem will be impossible when one of the two Circles falls wholly within the other, as well as when it falls wholly without it: But this Last depends upon the particular Order of joining the given Lines; whereas the Case is otherwise with respect to the first, or greatest Limit.

PROBLEM

PROBLEM XXXVII.

To divide a given Angle BAC into two Parts BAI and CAI, so that their Sines BE and CF may obtain a given Ratio; suppose that of m to n .

Put the Chord BC of the given Angle $= a$, and the



Part thereof BD, intercepted by AI, $= x$: Then, because of the similar Triangles BDE and CDF, it will be, BD (x) : CD ($a-x$) :: BE : CF :: m : n , by Hypothesis.

Therefore $nx = m \times a - x$; and consequently $x = \frac{ma}{m+n}$.

From whence, and the given Angle ABD, the Angle BAD will be found.

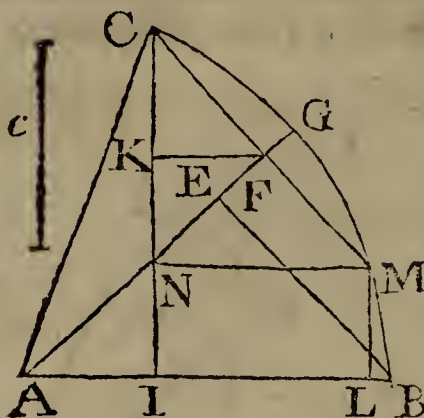
The Geometrical Construction (like the Algebraical Operation) may be performed by dividing the Subtense BC in the given Ratio of m to n : But the following Method is preferable.

In AC take $AG = m$; and, in BA produced, take $AH = n$; then a Line AE drawn parallel to That joining the Points G and H, will divide the Angle as required. For, by Trigonometry, AG (m) : AH (n) :: Sine AHG (BAE) : Sine AGH (CAE): From this Construction the numerical Solution is exceeding easy; both the Sides AG and AH, and the included Angle HAG being given.

PROBLEM

Geometrically.

If CI be supposed perpendicular to AB , and FK to CI ; then, the opposite Angles CNF and ANI being equal, their Complements NCF and NAI will likewise be equal: And therefore, the Triangles CFK and ABE being equiangular, we have $AB : BE :: CF : FK$, or $AB \times FK = BE \times CF = c^2$ (*according to Observation 5.*) whence FK is given. Therefore, if, from the Extremity of the Chord CFM , there be drawn MN perpendicular to CI , the Length thereof, being twice That of FK , will also be given; and, from thence, the following Construction.



Take IL a Third Proportional to $\frac{1}{2} AB$ and the Side (c) of the given Square, expressing the Magnitude of the proposed Rectangle: Draw LM perpendicular to AB , meeting the Arch BC in M ; then a Line AG drawn to bisect MC will divide the Angle BAC as required.

The Numerical Solution, from hence, is very concise and easy: For, having found the Value of IL (by dividing the Measure of the given Rectangle by Half the Radius) let the Co-sine Al of the whole, given, Angle be added thereto; then the Sum AL will be the Co-sine of (BM) the Difference of the two, required, Parts.— This Problem becomes impossible when IL is given greater than IB ; that is, when the Rectangle proposed is greater than Half the Rectangle under AB and BI .

PROBLEM

Geometrically.

If a Line DF be drawn to make the Angle BDF equal to CAE , the Triangles BDF and CAE will be similar; and we shall, therefore, have $AC \times BF = BD \times CE = c^2$ (according to *Observation 5. p. 88*) whence BF , and consequently AF , will be given. Moreover the Sum of the Angles BAD and BDF being equal to the given Angle BAC (by *Hypothesis*) and the same Sum $+ ADF$ equal to a Right-Angle (by *Elem. Cor. 4. to 10. 1.*) it is evident that ADF is equal to the Difference between the said given Angle BAC and a Right-One.

Therefore, having taken BF a Third Proportional to AB and the Side (c) of the given Square, make the Angle $AFO =$ the Angle FAC ; and from (O) the Intersection of AC and FO , let a Circle be described thro' A and F , intersecting the Tangent BK in D , and D ; from either of which Points draw AD , and the Thing is done.

In order to the Trigonometrical Calculation, let the Diameter GH be drawn to bisect the Arch AF in H , and let DM be perpendicular thereto: Then, having found AF , it will be, as $AN (\frac{1}{2} AF) : DM (NB) :: \text{Sine } AH (\text{Co-sine } NAO) : \text{Sine } GD (= \text{Co-sine } \frac{1}{2} DD) = \text{Co-sine of the Difference of the two required Angles } BAD \text{ and } DAI$; whence, as their Sum is given, the Angles themselves will be known.

This Problem becomes impossible, when the Circle $OAFG$ neither cuts, nor touches, the Line BK ; that is, when the given Rectangle is greater than the Square of the Tangent of Half the proposed Angle, and the Angle itself is acute; or, when the said Rectangle is less than the Square of half the Tangent of the proposed Angle, and the Angle itself is obtuse.

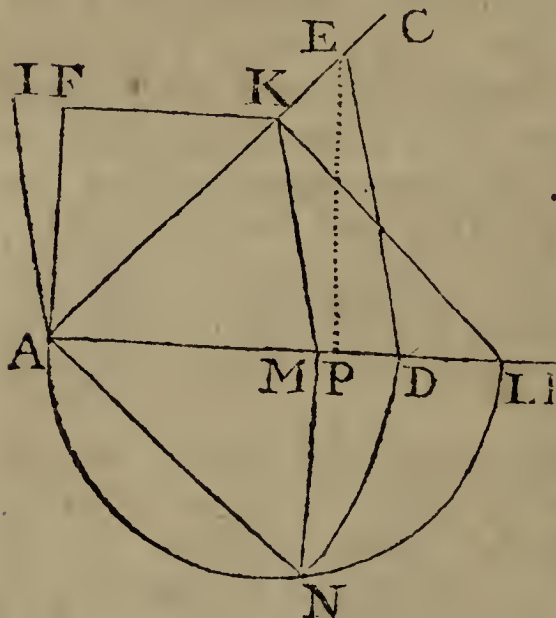
PROBLEM

PROBLEM XLI.

To draw a Line DE parallel to a given Line AI, so as to intersect two other Lines AB, AC, given by Position, and thereby form a Triangle ADE of a given Magnitude.

Let the given Area of the Triangle be denoted by

a^2 ; and let the Sines of the given Angles DAE, ADE, and AED (to the Radius r) be expressed by m , n , and p , respectively: Then, making EP perpendicular to AD, and calling AD, x ; we have (by plane Trigonometry) As $p : m :: x : DE = \frac{mx}{p}$; And $r : n :: DE : EP = \frac{mnx}{rp}$.



Hence $\frac{mnx}{rp} \times \frac{x}{2} = a^2 :$

And therefore $x = \sqrt{\frac{2rpa^2}{mn}} = a \sqrt{\frac{2rp}{mn}}.$

Geometrically.

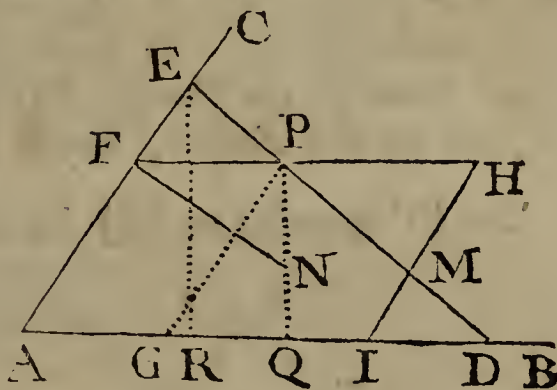
Let AF, perpendicular to AB, be the Side of a Square equal to the Triangle ADE; then, if AL be taken = 2AF, and FK be drawn parallel to AB, &c. it is evident that the Triangle ALK, being $= \overline{AF}^2$, will also be equal to the Triangle ADE. Moreover, by making KM parallel to AI, it will be $\overline{AD}^2 : \overline{AM}^2 :: ADE (ALK) : AMK$ (Elem. 17. 4.) :: AL : AM (Elem. 1. 4.) :: AL x AM : \overline{AM}^2 . And consequently $\overline{AD}^2 = AL \times AM$

x AM. Therefore, AD being a Mean Proportional between AL and AM, upon the former of These let a Semi-Circle ANL be described, intersecting MN, perpendicular to AB, in N; make AD = the Chord AN, and draw DE parallel to AI, and the Thing is done (*Elem.* II. 4).

P R O B L E M XLII.

Thro' a given Point P, between two Lines AB and AC, given by Position, to draw another Line DE, so that the Triangle ADE, formed thereby, may be of a given Magnitude.

Let PG and PF be parallel to AC and AB, and PQ and ER perpendicular to AB: Also let AG and PQ (which are given by the Position of P) be denoted by a and b , respectively. Then, calling AD, x , and denoting the given Area by c^2 , we



shall have, $DG (x-a) : PQ (b) :: AD (x) : ER = \frac{bx}{x-a} :$

And therefore $\frac{bx}{x-a} \times \frac{x}{2} (= ER \times \frac{1}{2} AD) = c^2$.

Whence $xx - \frac{2c^2x}{b} = -\frac{2ac^2}{b}$; and consequently $x =$

$$\frac{c^2 \pm c \sqrt{c^2 - 2ab}}{b}.$$

Geometrically.

If upon AF, a Parallelogram AFHI be constituted, to contain the given Area, it will appear, by taking away AFP MI from each of the equal Quantities AH and ADE, that the Remainders PHM and PFE + IDM will likewise be equal: And so, these three Triangles
L being

being all similar to one another, it follows that \overline{PH}^2 will be $= \overline{PF}^2 + \overline{ID}^2$ (*Elem.* 17. 4). Whence this Construction.

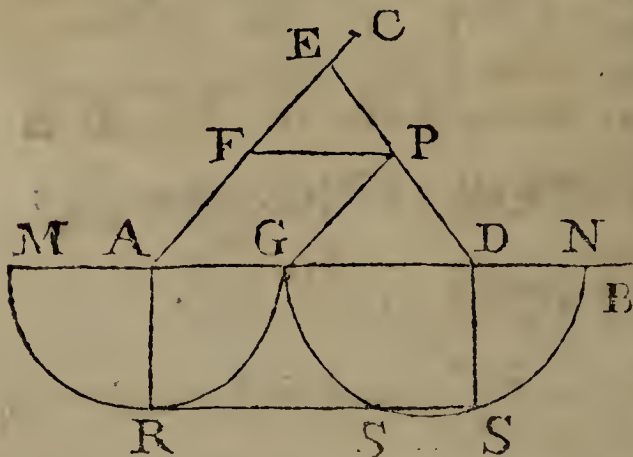
From the Center F, with the Interval PH, let an Arch be described, cutting PQ perpendicular to AB, in N; make $ID = PN$, and draw DPE, and the Thing is done. For it is evident that \overline{PH}^2 (\overline{FN}^2) is $= \overline{PF}^2 + \overline{ID}^2$ (\overline{PN}^2).—Hence it also appears that the Problem will be impossible when PH is less than PF; or when the Triangle proposed to be constructed is less than twice the Parallelogram AFPG.

PROBLEM XLIII.

Thro' a given Point P, between two Lines AB, AC given by Position, to draw another Line BC, so that the Sum of the Parts, AD + AE, cut off thereby, from the two Former, may be a given Quantity.

Let PF (parallel to AB) be denoted by a , and PG

(parallel to AC) by b ; also let $AD + AE = c$, and $AD = x$: Then, by Reason of the parallel Lines, we shall have $DG (x - a) : PG (b) :: AD (x) : AE = \frac{bx}{x - a}$.



And therefore $\frac{bx}{x - a} + x (= AE + AD) = c$.

From which, by Reduction, $xx - c + a - b \times x = -ac$;

and consequently $x = \frac{c + a - b}{2} + \sqrt{\frac{c + a - b^2}{4} - ac}$.

Geometrically.

Geometrically.

The Triangles DGP and PFE being similar, the Rectangle under DG and FE will therefore be equal to the given Rectangle under AF and $AG = \overline{AR}^2$; by taking $AM = AF$, and making AR a Mean Proportional between AM and AG. Moreover, if MN be taken = the given Sum of AD and AE, it is manifest that the Sum of DG and FE (whose Rectangle is above given) will, also, be given = GN.

Therefore, having described a Semi-circle upon GN, let RS be drawn parallel to AB, intersecting the Periphery thereof in S; from which Point, upon AB, let fall the Perpendicular SD, and thro' P draw DPE, and the Thing is done. For $DG \times DN = \overline{DS}^2 = \overline{AR}^2 = AM \times AG = AF \times AG = DG \times EF$: Whence $EF = DN$; and consequently $AD + AE (= AD + AF + DN) = MN$.

It is plain, this Problem becomes impossible, when RS falls below the Semi-circle GN; or, when the Sum of AD and AE is supposed to exceed That of AG and AF by less than the Double of a Mean Proportional between AG and AF.

Much after the same Manner the Problem will be resolved, when the Difference, the Ratio, or the Rectangle of AD and AE is given.

As, in the present Case, the Line AP is given, both in Length and Position, let the Rectangle under it, and a Part, PQ, of the same Line produced, be therefore assumed $= BP \times CP$; then the Consequence will be, that, besides obtaining $PQ = \frac{d^2}{AP}$, the Triangles PQC and PBA (supposing QC drawn) will also be similar (because $AP : BP :: CP : PQ$) And so, the Angle PCQ being $=$ the given Angle PAB, it is evident that a Segment of a Circle described upon PQ (by *Elem.* 17. 5.) capable of containing the said given Angle, will intersect AE in the Point (or Points) required.

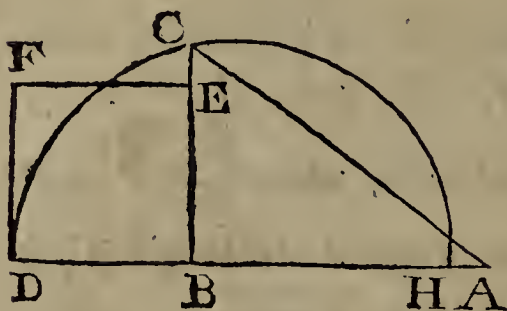
This Problem will, it is manifest, be impossible, when the Circle, described as above, falls short of AE; or, according to the Algebraic Solution, when $\left[\frac{dd}{2b} + c \right]^2 - a^2$ is negative; that is, when the proposed Rectangle is less than $2b \times \overline{a-c}$, or, its Equal $2PH \times \overline{PF-FG}$.

PROBLEM XLV.

The Area (a^2) of a right-angled Triangle ABC, whose Sides are in Arithmetical Progression, being given, to determine the Triangle.

Put the greater Leg $AB = x$, and the Common Difference $= y$; so shall $BC = x - y$, and $AC = x + y$. And therefore

$$\left\{ \begin{array}{l} (x+y)^2 = (x-y)^2 + x^2 \\ \frac{1}{2} x \times (x-y) = a^2 \end{array} \right\} \text{by the Question.}$$



From the former of which Equations we have $xx + 2xy + yy = 2xx - 2xy + yy$; and consequently $4y = x$: Whence, by substituting for x in the second Equation, we get $2y \times 3y = a^2$: From

which y is given $= \sqrt{\frac{a^2}{6}}$; and $x (=4y) = 4\sqrt{\frac{a^2}{6}}$.

Therefore $BC = 3\sqrt{\frac{a^2}{6}}$, $AB = 4\sqrt{\frac{a^2}{6}}$, and

$AC = 5\sqrt{\frac{a^2}{6}}$.

Geometrically.

It is well known that $\overline{AC+BC} \times \overline{AC-BC} (= \overline{AC}^2 - \overline{BC}^2)$ is $= \overline{AB}^2$ (*Elem. Corol. to 7. 2*). And, by the Question, $AC + BC$ is $= 2AB$ (because AB is an Arithmetical Mean between AC and BC) Therefore $2AB \times \overline{AC-BC} = \overline{AB}^2$; and consequently $AC-BC = \frac{1}{2} AB$: Take these equal Quantities from the equal Quantities $AC+BC$ and $2AB$, and the Remainders, $2BC$ and $\frac{1}{2} AB$, will be equal; and consequently $3AB = 4BC$. But, $\frac{1}{2} AB \times BC (= \frac{1}{3} BC \times BC) = \overline{BD}^2 (= a^2)$; and therefore $\overline{BC}^2 = \frac{3}{2} \overline{BD}^2$.

Hence

Hence, if BH be taken to BD (a) in the Proportion of 3 to 2, a Mean Proportional BC between DB and HB, will be the lesser Leg of the Triangle required; whose greater Leg BA, being in Proportion thereto, as 4 to 3, is also given from hence.

P R O B L E M XLVI.

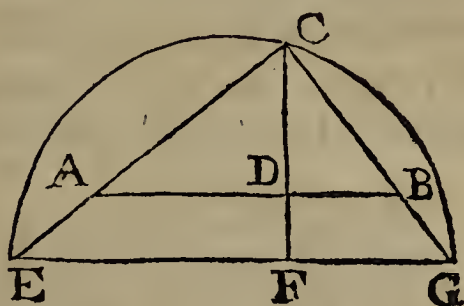
The Area (a^2) of a right-angled Triangle ABC, whose Sides are in Geometrical Proportion, being given; to determine the Triangle.

Make $AC = x$; then $BC = \frac{2aa}{x}$, and $AB =$

$$\sqrt{x^2 + \frac{4a^4}{xx}}.$$

Therefore, $\frac{2a^2}{x} : x :: x :$

$$\sqrt{x^2 + \frac{4a^4}{xx}}, \text{ by the Quest.}$$



$$\text{Hence } x^2 = \frac{2a^2}{x} \sqrt{x^2 + \frac{4a^4}{xx}}; \text{ and } x^4 = \frac{4a^4}{xx} \times \frac{x^4 + 4a^4}{xx};$$

$$x^8 - 4a^4x^4 = 16a^8; \text{ and } x^4 - 2a^4 = a^4 \sqrt{20}; \text{ and } x = a \times$$

$$\sqrt[4]{2 + \sqrt{20}}.$$

Geometrically.

Since, by Hypothesis, $AB : AC :: AC : BC$, therefore is $\overline{AB}^2 : \overline{AC}^2 :: \overline{AC}^2 : \overline{BC}^2$ (*Elem. Cor. 1. to 9. 4.*)

But \overline{AC}^2 , supposing CD perpendicular to AB, is equal to $AB \times AD$; and \overline{BC}^2 equal to $AB \times BD$ (*Elem. Cor. to 11. 4.*).

Therefore $\overline{AB}^2 : AB \times AD :: AB \times AD : AB \times BD$; or, $AB : AD :: AD : BD$.

Whence the following Construction.

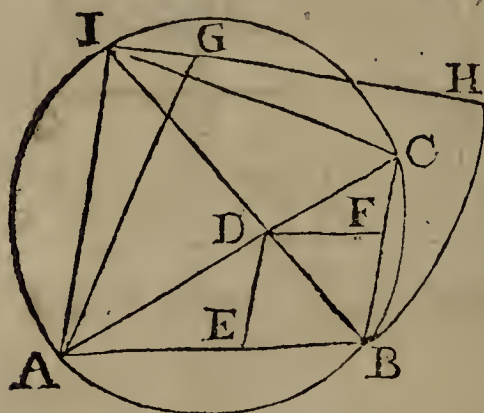
Draw any Line EG, at pleasure; which divide at F (*by Elem. 22. 5.*) according to Extreme-and-Mean Proportion (so that $EG : EF :: EF : GF$); erect the per-

pendicular FC, and upon EG let a Semi-circle be described intersecting it in C; join E, C, and G, C; and let AB (by Prob. 41.) be drawn, parallel to EG, to cut off the given Area ABC ($=a^2$), and the Thing is done. For it is manifest that AB is divided, by CDF, in the same Proportion with EG; or that, $AB : AD :: AD : BD$, as above.

P R O B L E M XLVII.

Supposing one Side AC, and the opposite Angle Angle ABC of a Triangle to be given, together with the Side DE, or DF, of the inscribed Rhombus EF; to find from thence the other two Sides of the Triangle.

Conceive a Circle ABCI to be described about the Triangle, and the Diagonal of the Rhombus produced to meet the Circumference thereof in I: Also



let AI and CI be drawn; which will be equal to each other (*Elem.* 10. 3.) as being the Subtenses of the equal Angles IBA and IBC, form'd by the Diagonal and Sides of the Rhombus: And, since the Arcs AI and CI, as well as the Chords, are equal, the Angles IAD and

IBA, insisting upon Them, must likewise be equal; and consequently the Triangle IAD similar to the Triangle IBA.

But the vertical Angle AIC of the Isosceles Triangle ACI (as well as the Base AC) is given, being $= 2$ Right-angles $- ABC$ (*Elem.* 13. 3.) whence IA will be given (*by plane Trigonometry*).

Put, therefore, $IA=a$, $BD=b$, and $IB=x$; then, from the Similarity of the Triangle above specified, we shall have, $x-b$ (ID) : a (IA) :: a (IA) : x (IB) whence $xx-bx=aa$; and consequently $x=\sqrt{aa+\frac{1}{4}bb}+\frac{1}{2}b$. From which, and the known Values of IA and the Angle ABI, the Value of AB, &c. will also be known.

Geometrically.

Geometrically.

The Geometrical Construction hereof, as the Rectangle $BI \times DI$ (by the foregoing Proportion) is given $= \overline{AI}^2$, is obvious from Problem 6; and is thus,

Having, upon the given Side AC, described a Segment of a Circle ABC capable of containing the given Angle ABC (*Elem.* 17. 5); and, in the other Segment, AIC, constituted the Isosceles Triangle AIC, make IG perpendicular to IA, and equal to Half (BD) the given Diagonal of the Rhombus; also, in IG produced, take GH equal to the Distance GA; and, through H, from the Center I, let an Arch be described intersecting the Circle ABC in B; then draw BA and BC, and the Thing is done.

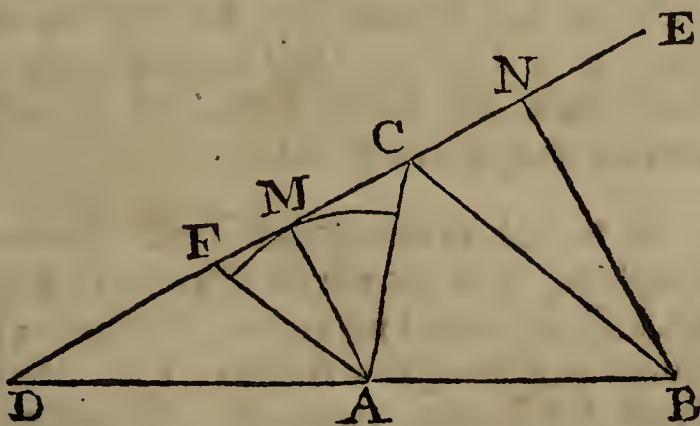
PROBLEM XLVIII.

From two given Points A and B, to draw two Lines AC and BC, to meet in a Right-line DE given by Position, and form an Angle ACB of a given Magnitude.

Make AM and BN perpendicular to DE ; and let AF be supposed parallel to BC :

Then, calling
AM, a ; BN,
 b ; MN, c ; and
MC, x ; it will
be as $b : c - x$
(NC) :: $a : MF$

$$\frac{ac - ax}{b} :$$



But MF and MC are Tangents of the Angles MAF and MAC; which Angles together (because of the parallel Lines AF and BC) make an Angle CAF equal to the given Angle ACB: Therefore, if the Tangent of the said given Angle, to the Radius AM, be denoted

noted by t , it will appear, from Problem 25, that

$$\frac{MC + MF \times AM^2}{AM^2 - MC \times MF} = t.$$

This, in Species, gives
$$\frac{x + \frac{ac - ax}{b} \times a^2}{a^2 - \frac{acx - axx}{b}} = \frac{bx - ax + ac \times a}{ab - cx + xx} = t.$$

Which, by Reduction, becomes
$$\frac{b - a \times ax}{t} + \frac{aac}{t} = xx - cx + ab.$$
 Whence, making $c + \frac{b - a \times a}{t} = d$,

and $a \times \frac{ac}{t} - b = f$, we have $f = xx - dx$; and con-

sequently $x = \frac{1}{2} d \pm \sqrt{f + \frac{1}{4} dd}.$

When the Line MN is parallel to That joining the given Points A, B; then, BN (b) becoming = AM (a) we have, in this Case, $d = c$, and $f = \frac{aac}{t} - a^2$; and

therefore $x = \frac{c}{2} \pm \sqrt{\frac{aac}{t} - aa + \frac{1}{4} cc}.$ Which may

serve as a Theorem for finding the Segments of the Base of a Triangle (and consequently the Triangle itself) when the whole Base, the Perpendicular, and the vertical Angle are given.

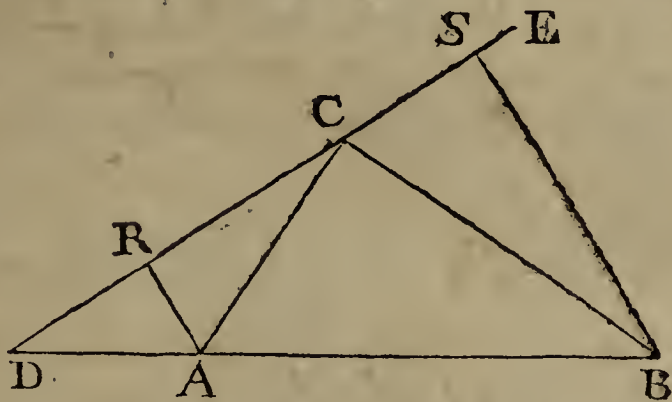
As to the Geometrical Construction of the General Problem, it is extremely obvious; since a Segment of a Circle described upon AB (by *Elem.* 17. 5.) capable of containing the given Angle, will intersect DE in the Point (or Points) required. Whence it also appears that the Problem will be impossible when the Circle falls short of the Line FE; and, consequently, that the Angle ACB will be the greatest possible when the Circle touches the said Line; or, when DC is a Mean Proportional between DA and DB (*Elem. Cor. to 17. 3*).

PROBLEM

PROBLEM XLIX.

To find a Point C in a Right-line DE, given by Position; so that two Lines CA, CB being drawn from thence to two given Points A and B, the One BC shall exceed the Other AC by a given Difference d.

Let AR and BS be perpendicular to DE; and let these Perpendiculars, together with RS (which are all known from the Position of DE) be denoted by a , b , and c , respectively: Then,



putting $AC = x$, we have $RC \left(= \sqrt{AC^2 - AR^2} \right)$

$$= \sqrt{xx - aa}; \text{ and } SC \left(= \sqrt{BC^2 - BS^2} \right) = \sqrt{xx + 2xd + dd - bb} = c - \sqrt{xx - aa}.$$

From which Equation, by squaring both Sides thereof, we get $xx + 2dx + dd - bb = cc - 2c\sqrt{xx - aa} + xx - aa$.

This contracted becomes $2c\sqrt{xx - aa} = bb - aa - dd + cc - 2dx$, or, $\sqrt{xx - aa} = m - nx$; by dividing by $2c$, and putting $\frac{bb - aa - dd + cc}{2c} = m$, and $\frac{d}{c} = n$.

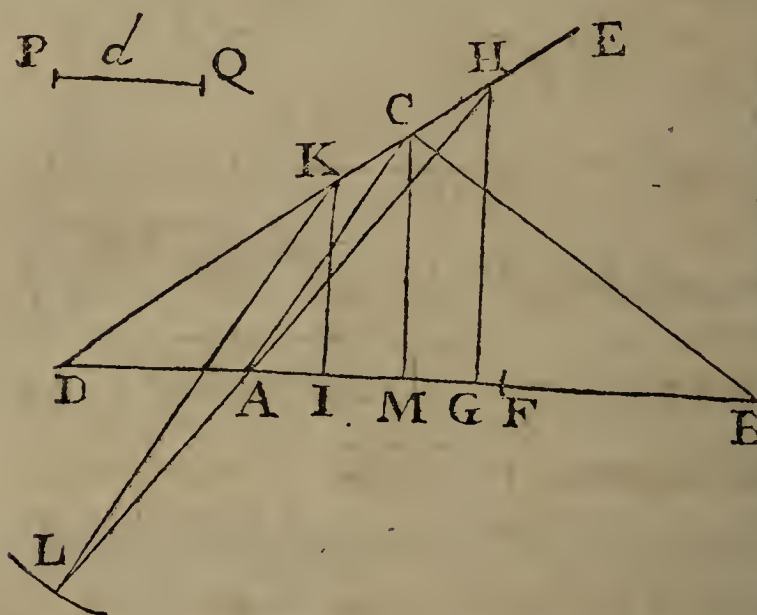
Therefore, by squaring again, $xx - aa = mm - 2mnx + nnxx$; whence $1 - nn \times xx + 2mnx = aa + mm$, or $xx + \frac{2mnx}{1 - nn} = \frac{aa + mm}{1 - nn}$.

$$\text{And consequently } x = \sqrt{\frac{aa + mm}{1 - nn} + \frac{m^2 n^2}{(1 - nn)^2} - \frac{mn}{1 - nn}}.$$

Geometrically,

Geometrically.

If AB be drawn, and bisected in F, and CM be supposed perpendicular thereto, it is well known that $FM = \frac{BC + AC \times BC - AC}{2AB}$ (*Elem. 8. 2.*): Therefore, since $BC - AC = PQ$, and, consequently, $BC + AC = PQ + 2AC$, it follows that $FM = \frac{PQ + 2AC \times PQ}{2AB} = \frac{PQ^2}{2AB} + \frac{AC \times PQ}{AB}$.



But, since both AB and PQ are given, the Value of

$\frac{PQ^2}{2AB}$, (being a Third Proportional to $2AB$ and PQ) is also given.

Let this, therefore, be expressed by

FG ; so that FM may be $= FG + \frac{AC \times PQ}{AB}$: Then

it is plain that $GM = \frac{AC \times PQ}{AB}$; or that, GM is to

AC in the given Ratio of PQ to AB .

But the Ratio of GM to HC (supposing GH perpendicular to AB) is also given by the Position of the Line DE ; whence the Ratio of HC to AC (where Both belong to the same Triangle AHC) will be exhibited: For, taking $GI = PQ$, and making IK perpendicular to AB , it will be $\frac{HC}{HK} = \frac{GM}{GI} (= \frac{GM}{PQ}) = \frac{AC}{AB}$,

(*p. above*) and consequently $HC : AC :: HK : AB$.

But,

But, if KL be supposed parallel to AC, meeting HA produced (if need be) in L, it is manifest that $HC : AC :: HK : KL$. Therefore $KL = AB$; whence the following Construction.

Having taken $FG = \frac{\overline{PQ}^2}{2AB}$, $GI = PQ$, and erected the Perpendiculars GH and IK, and also drawn HA (as above specified), from the Center K, with the Interval AB, let an Arch be described, cutting HA produced in L; and, having drawn LK, make AC parallel thereto; which will cut the given Line DE in the Point required.

The Trigonometrical Calculation, from this Construction, may be as follows.

Having computed $FG \left(= \frac{\overline{PQ}^2}{2AB} \right)$ and subtracted it from FD and FA, the Remainders GD and GA will be given; and it will then be as $GD : GA :: \text{Tang. DHG (or Co-tang. D) : Tang. AHG}$; whence the Angle LHK is likewise known. And, since $\text{Sin. DHG (or Co-sin. D) : Rad.} :: IG (PQ) : HK$, the Value of HK (as well as Those of KL and the Angle HKL) will be known: From which, the other two Angles of the Triangle HKL being found, the Angle DAC (= ECA - D = HKL - D) will also be obtained.

After the very same Manner the Problem may be resolved, when the Sum, instead of the Difference, of the Lines AC and BC is given.

As to the Restrictions of the last Problem, it is evident that the given Difference must never exceed the Distance AB: But when the Line DE passeth between the Points A and B, the Limit will be still less; but is easily determined, in any Case, from the given Position of DE. It is a little remarkable, that the above Solution fails in that particular Case, only, wherein the General Problem becomes most simple; that is, when DE is perpendicular to AB. But here the Operation will, also, become more simple and expeditious: For the

$$\frac{AC \times \overline{BD}^2}{AB} = AC \times BC + AC \times BG \text{ (by taking } BG = \frac{\overline{BD}^2}{AB} \text{)}$$

$$= AC \times \overline{BC + BG} = AC \times CG.$$

Therefore, having taken BG a Third Proportional to AB and BD; and CL a Mean-Proportional between AC and CG; draw BK perpendicular to GF, meeting the Circumference of a Semi-circle, described upon AC, in K; and, having drawn AKM, and taken AH equal to the given Difference of AD and CD, upon H as a Center, with the Radius LC, let an Arch be described, intersecting AK in M; from which Point upon GF let fall a Perpendicular MN; then, if from A, with the Radius AN, another Arch be described, it will intersect the Arch of the given Semi circle in the Point, D, required.

For $AB : BC :: \overline{AB}^2 : AB \times BC (= \overline{BK}^2)$, *Elem.*

II. 4 $:: \overline{AN}^2 (\overline{AD}^2) : \overline{NM}^2 = \overline{AD}^2 \times \frac{BC}{AB}$. And

$$\overline{NM}^2 \left(\overline{AD}^2 \times \frac{BC}{AB} \right) + \overline{HN}^2 = \overline{HM}^2 = AC \times CG$$

(by Constr.) $= \overline{AD}^2 \times \frac{BC}{AB} + \overline{CD}^2$ (p. above):

Therefore $CD = HN = AN + AH = AD + AH$; and consequently $CD - AD = AH =$ the given Difference, *by Construction*.

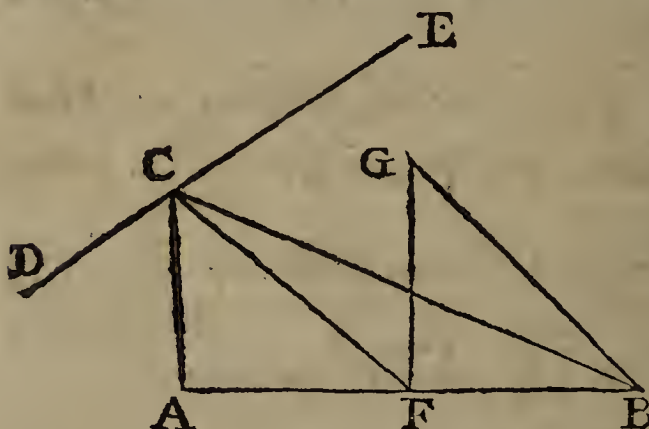
The Method of Calculation, from this Construction, is sufficiently easy: For, having computed $BG (= \frac{\overline{BD}^2}{AB})$ and $HM (= \sqrt{CA \times CB + BG})$ and also the Angle BAK (which is had by the Proportion $AB : BC :: \text{Squ. Rad.} : \text{Squ. Tang. BAK}$); you will then have, in the Triangle HAM, two Sides and one Angle; whence every-thing else is readily determined.

PROBLEM

PROBLEM LI.

From the Middle F , and the two Extremes A and B , of a given Right-line AB , to draw three Lines to meet in a Point C , in a Right-line DE given by Position, so as to be in Geometrical Proportion, or so that $AC : FC :: FC : BC$.

Because $\overline{BC}^2 + \overline{AC}^2 = 2\overline{FC}^2 + 2\overline{BF}^2$ (*Elem.* 12.2). And $BC \times AC = \overline{FC}^2$ (by *Hyp.* and *Elem.* 3. 4.) it is evident that $\overline{BC}^2 - 2BC \times AC + \overline{AC}^2 = 2\overline{BF}^2$, or $\overline{BC-AC}^2 = 2\overline{BF}^2 = \overline{BG}^2$, by taking BG as the Diagonal of the Square whose Side is BF .



Hence $BC-AC$ is given $= BG$: And so the Case in Question is reduced to Problem 49; to which I shall therefore refer for the remaining Part of the Solution.—It ap-

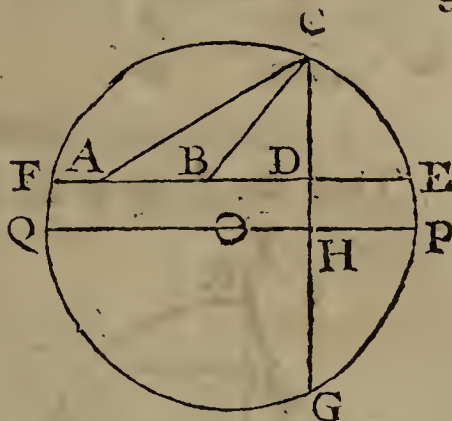
pears from hence that a Point in the Circumference of a given Circle, whose Center is in the Line AB , may be so determined, by the last Problem, that three Lines drawn from thence to the three given Points A , F , B , shall be in Geometrical Proportion.

PROBLEM

PROBLEM LII.

From two given Points A, B, within a given Circle, to draw two Lines AC, BC, to meet in the Periphery Thereof, so that the Sum of their Squares may be a given Quantity.

Through the given Points let EF be drawn, meeting the Circumference of the Circle in E and F; parallel to which, draw the Diameter PQ; and let the Chord CG be drawn to cut EF and PQ, at Right-angles, in D and H.



Put $EF = a$, $EA = b$,
 $EB = c$, $DH = d$, $ED = x$;

and $DC = y$; and let the given Quantity, $\overline{AC}^2 + \overline{BC}^2$, be denoted by e^2 : Then will $DF = a - x$, $DA = b - x$, $DB = c - x$, $DG = y + 2d$.

But $ED \times DF = CD \times DG$ (*Elem.* 16. 3).

And $\overline{DA}^2 + \overline{DC}^2 + \overline{DB}^2 + \overline{DC}^2 = \overline{AC}^2 + \overline{BC}^2$ (*Elem.* 7. 2). Which, in Species,

$$\text{give } \begin{cases} x \times a - x = y \times y + 2d \\ \overline{b-x}^2 + y^2 + \overline{c-x}^2 + y^2 = e^2. \end{cases}$$

$$\text{or } \begin{cases} ax - xx = yy + 2dy \\ 2xx + 2yy - 2bx - 2cx = ee - bb - cc. \end{cases}$$

Whence, by adding the double of the former Equation to the Latter, we get $2ax - 2bx - 2cx = ee - bb - cc + 4dy$;

and consequently $y = \frac{bb + cc - ee}{4d} + \frac{a - b - c}{2d} \times x = f +$

gx , by making $\frac{bb + cc - ee}{4d} = f$, and $\frac{a - b - c}{2d} = g$.

M

Now,

Now, the Value of y thus found being substituted in the first Equation, there arises $ax - xx = ff + 2fgx + g^2x^2 + 2df + 2dgx$;

$$\text{or, } 1 + gg \times xx + 2dg + 2fg - a \times x = -ff - 2df.$$

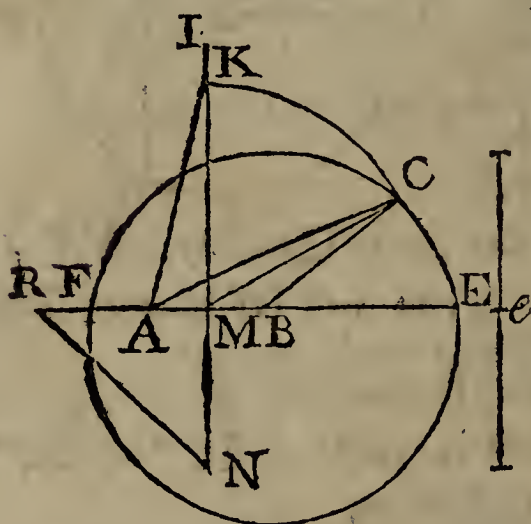
From which, by making $m = 1 + gg$, $n = a - 2dg - 2fg$,

and $p = f \times \overline{f + 2d}$, the Value of x is found $= \frac{n}{2m} \pm$

$$\sqrt{\frac{m}{4mm} - \frac{p}{m}}.$$

Geometrically.

That the *Locus* of the Vertex of a Triangle, whereof the Base and Sum of the Squares of its 2 Sides are given, is the Circumference of a Circle, described from the Middle of the Base as a Center, is evident, from *Elem.* 12. 2; because the Line drawn from the Vertex to the Middle of the Base, is an invariable Quantity.



Therefore, having bisected AB with the Perpendicular IMN, take MN and MR equal, each, to Half the Side (e) of the given Square; draw NR, and, from A to the Perpendicular MI, draw AK equal to NR; and upon the Center M, at the Distance MK, let an Arch be described; which will meet the Circumference of the given Circle in the Point C, required. For AC, BC, and MC being drawn, it is evident that $\overline{AC}^2 + \overline{BC}^2 = 2\overline{AM}^2 + 2\overline{CM}^2 = 2\overline{AM}^2 + 2\overline{MK}^2 = 2\overline{AK}^2 = 2\overline{RN}^2 = 4\overline{RM}^2 = e^2$. In this Problem it is requisite that MC ($= \sqrt{\frac{1}{2}e^2 - \overline{AM}^2}$) should be greater than the Difference, and less than the Sum, of the Radius

Radius of the given Circle and the Distance of the Point M from its Center.

PROBLEM LIII.

To draw a Line PQ thro' a given Point A, so that the Rectangle of two Perpendiculars, falling thereon from two other given Points, B and C, may be of a given Magnitude.

Let BA and CQ be produced to meet in E; and upon BE let fall the Perpendicular CF; which, and the Segment AF, are both given, by the Position of the proposed Points.

Put, therefore, CF = a, AF = b, AB = c, and EF = x: Then

will $CE = \sqrt{aa + xx}$:

And it will be (by similar Triangles),

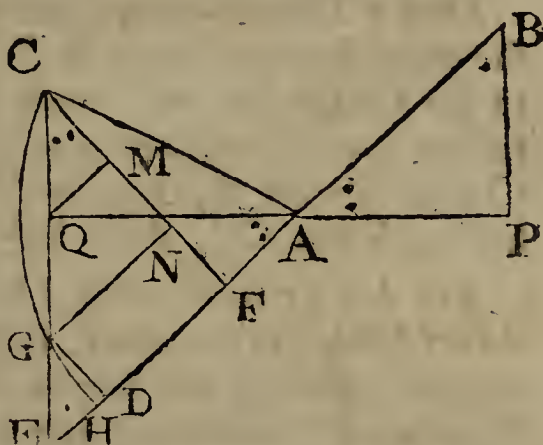
$$\text{as } CE : EF :: AB : BP :: \frac{cx}{\sqrt{aa + xx}};$$

$$\text{and } CE : EF :: AE : EQ = -\frac{xx \ b + x}{\sqrt{aa + xx}};$$

$$\text{Whence } CQ (= CE - EQ) = \sqrt{aa + xx} - \frac{bx + xx}{\sqrt{aa + xx}} = \frac{aa - bx}{\sqrt{aa + xx}};$$

And, consequently, $BP \times CQ = \frac{cx \times aa - bx}{aa + xx} = g^2$; supposing g^2 to denote the given Magnitude of the proposed Rectangle.

From this Equation, by making $d = \frac{aac}{bc + gg}$, and $f =$



$\frac{aag}{bc+gg}$, the Value of x is found $= \frac{1}{2} d \pm \sqrt{\frac{1}{4} dd - fg}$.

Whence every Thing else is readily determined.

Geometrically.

If QM be supposed parallel to AB, the Triangle CQM will, it is plain, be similar to ABP; and consequently $AB \times QM = BP \times CQ$ (*Elem.* 18. 3.) $= g^2$: Whence QM is given; and, from thence, the following Construction.

Find a Third-Proportional to AB and the Side (g) of the given Square; and, in BF produced, take FD equal to the Double thereof; draw DG parallel to FC, intersecting the Circumference of a Circle, described from the Center A with the Radius AC, in the Point G; and, having drawn CG, draw PQ perpendicular thereto.

For the Trigonometrical Calculation, it will be $AF : AD (AF + FD) :: \text{Co-f. HGC} : \text{Co-f. HG}$, the Difference between the Angles BAP (HAQ) and CAQ: From which, as their Sum is given, the Angles themselves will be known.—This Problem becomes impossible when FD is greater than FH; that is, when the proposed Rectangle is greater than Half the Rectangle under AB and FH.

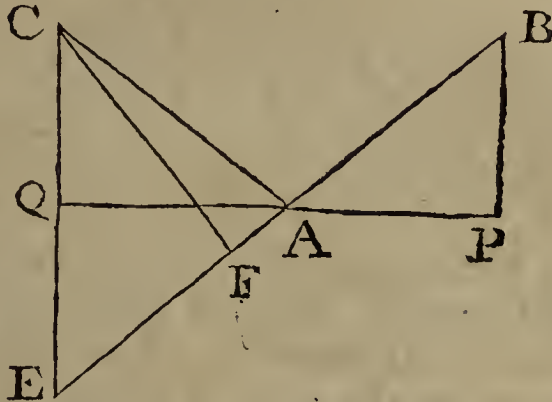
PROBLEM

PROBLEM LIV.

Two Lines AB, AC, drawn from the same Point A, being given both in Position and Length; to draw another Line PQ thro' that Point, so that two Perpendiculars BP, CQ, falling thereon from the Extremes of the two given Lines, may form two Triangles ABP, ACQ, equal to each other.

Let BA and CQ be produced to meet in E; and upon BE let fall the Perpendicular CF: And put $CF = a$, $AF = b$, $AB = c$, and $EF = x$.

The Areas of similar Triangles being, in Proportion, as the Squares of their homologous Sides; and both the Triangles AEQ, ABP



being similar to ECF; we have $\overline{CE}^2 (aa + xx) : \frac{1}{2} CF \times EF (\frac{1}{2} ax) :: \overline{AB}^2 (c^2) : \text{Area ABP} = \frac{\frac{1}{2} ac^2 x}{aa + xx}$.

And, $aa + xx : \frac{1}{2} ax :: \overline{x + b}^2 (= \overline{AE}^2) : \text{Area AEQ} = \frac{\frac{1}{2} ax \times \overline{x + b}^2}{aa + xx}$. This, taken from the Area AEC (=

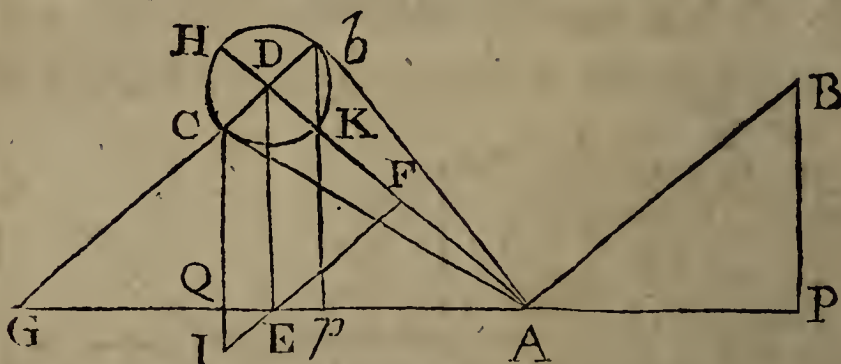
$\frac{1}{2} a \times \overline{x + b}$) leaves the Area ACQ = $\frac{\frac{1}{2} a \times \overline{x + b} \times aa - bx}{aa + xx}$.

Which being equal to ABP, by Hypothesis, we therefore have $\overline{x + b} \times aa - bx = c^2 x$: From whence, making

$d = \frac{cc - aa}{b} + b$, x is found = $\sqrt{aa + \frac{1}{4} dd} - \frac{1}{2} d$.

Geometrically.

If Ab be supposed perpendicular, and equal, to AB , and bp perpendicular to PQ , the Triangle Abp , being similar to ABP , will also be equal to it; and consequently equal to ACQ : And, if these equal Triangles be, successively, taken from the Trapezium $AQCb$, the Remainders pb CQ , and ACb will likewise be equal.



Hence it appears that the Parallelogram under Cb and DE , whose Angle is CDE (supposing DE parallel to, and an Arithmetical Mean between, CQ and bp) is equal to the given Triangle ACb ; and consequently the Altitude Thereof equal to half That of the Triangle. Whence it is evident that the Point E must fall, somewhere, in a Line FI drawn thro' the Middle of AD (or AC) parallel to bC (*Elem.* 2. 2). But the Point E , since the Angle ADE is a Right-one, will likewise fall in the Circumference of a Semi-circle described upon the Diameter AD (*Elem.* 11. 3). And therefore FE , being a Radius, must be equal to AF ; and consequently $DG = AD$; supposing DC produced to meet AQ in G .

Therefore, in order to the Geometrical Construction, having made Ab perpendicular, and equal to, AB , and drawn AD to the middle of Cb (as above intimated) let DG , in DC produced, be taken equal to AD ; and from G , through A , draw GP , and the Thing is done.

It often happens that the Demonstration of a Geometrical Construction, to be the most neat and elegant, proceeds upon Principles very different from Those whereby we first arrived at such Construction. The
Case

Case above is an Instance of it: Where, from the similar Triangles, it is manifest that $GQ (Ap) : QC :: Gp (AQ) : pb$; and therefore $\frac{1}{2} AQ \times QC = \frac{1}{2} Ap \times pb = \frac{1}{2} BP \times AP$. Q. E. D.

Note. If AB and AC be two Semi-conjugate Diameters of an Ellipsis, then the Line PAQ, determined as above, will be the Position of the Greater Axis: And, if, upon the Diameter Cb, a Circle be described intersecting AD in H and K; then AH, and AK will be equal in Length to the greater, and lesser, Semi-Axis, respectively.

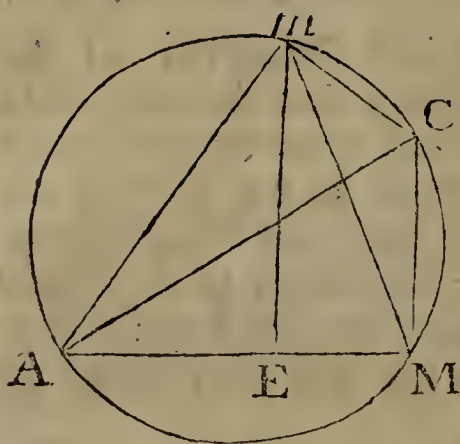
From whence the most useful Properties of the Conjugate Diameters of an Ellipsis may be very easily deduced. Such, as that, the Sum of the Squares of any two Conjugate Diameters, is equal to the Sum of the Squares of the two Axes: And that, any Parallelogram described about the Conjugate Diameters of an Ellipsis is equal to the Rectangle under the two Axes; and so forth. But these are Matters not altogether proper to be insisted on in this Place.

However it will not be improper to observe, that the last Problem is always possible, except in those two Cases, wherein the given Lines are perpendicular, and form one continued Line.

PROBLEM LV.

If, at the Extremes, M and m , of two given Right-lines AM and Am , making a given Angle MAM , two Perpendiculars MC and mC be erected; 'tis proposed to find the Distance of their Intersection, C , from A the given angular Point.

Since the Angles AMC and AmC are both Right Ones, the Circumference of a Circle, described upon the



Diameter AC , will pass thro' M and m . Therefore, if Mm be drawn, and a Perpendicular mE be let fall upon AM , the Triangles AmC and mEM will appear to be equiangular; because the Angles ACm and EMm , insisting on the same Arch Am , are equal;

and AmC is equal to mEM , being both Right Angles.

If now the Ratio of mE to Am (which is given because the Angle EAm is given) be denoted by That of s to r ; and the Ratio of AE to Am by That of c to r ;

we shall have $AE = \frac{c}{r} \times Am$; and therefore $\overline{Mm}^2 (= \overline{AM}^2 + \overline{Am}^2 - 2AM \times AE) = \overline{AM}^2 + \overline{Am}^2 - \frac{2c}{r} \times AM \times Am$: Whence, by Reason of the similar

Triangles above specified, it will be, $\overline{mE}^2 : \overline{mA}^2 (:: s^2 : r^2) :: \overline{Mm}^2 : \overline{AC}^2 = \frac{r^2}{s^2} \times \overline{AM}^2 + \overline{Am}^2 - \frac{2rc}{ss} \times$

$AM \times Am$; whence AC is given.

As to the Geometrical Construction, it is indicated by the Conditions of the Problem, without any sort of Argumentation.

PROBLEM

PROBLEM LVI.

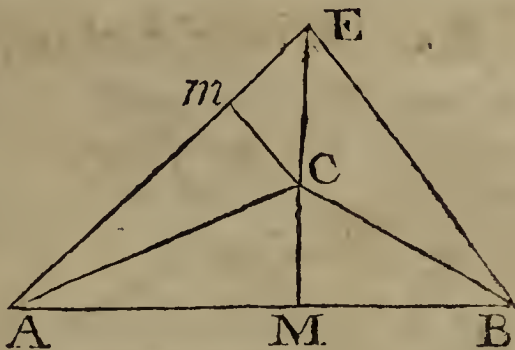
To find a Point C, from whence three Right-lines drawn to so many given Points A, B, and E, shall obtain the Ratio of three given Quantities a , b , and c , respectively.

The given Points being joined ; make $AB = f$, $AE = g$, and $AC = x$: Then,

BC being $= \frac{bx}{a}$, and EC

$= \frac{cx}{a}$ (by Hypothesis) we

also have $AM (= \frac{1}{2} AB + \frac{AC^2 - BC^2}{2AB}) = \frac{f}{2} +$



$\frac{aa - bb \times x^2}{2aaf}$ (Elem. 8. 2.) ; and $Am (= \frac{1}{2} AE +$

$\frac{AC^2 - EC^2}{2AE}) = \frac{g}{2} + \frac{aa - ee \times x^2}{2aag}$; supposing CM and

Cm to be perpendicular to AB and AE , respectively.

Hence, putting $\frac{aaf}{aa - bb} = h$, $\frac{aag}{aa - ee} = k$, and $z =$

$\frac{f}{2} + \frac{aa - bb \times x^2}{2aaf}$ ($= AM$) $= \frac{f}{2} + \frac{xx}{2h}$, we get $xx =$

$2hz - fh$; and therefore $Am (= \frac{g}{2} + \frac{xx}{2k}) = \frac{g}{2} +$

$\frac{2hz - fh}{2k} = l + \frac{hz}{k}$; by making $l = \frac{g}{2} - \frac{fh}{2k}$.

Now, since (by the last Problem) $\overline{AM}^2 + \overline{Am}^2 =$

$\frac{2c}{r} \times AM \times Am = \frac{ss}{rr} \times \overline{AC}^2$ (where s and c denote

the Sine and Co-sine of the given Angle MAm , to the Radius r) we shall, from hence, by substituting the
above

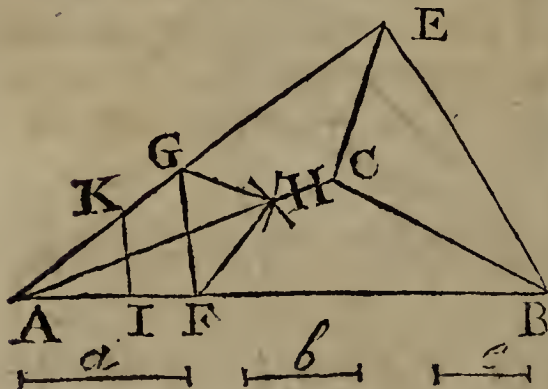
above Values of AM , Am , and AC , obtain the following Equation,

$$z^2 + l + \frac{bz}{k} \Bigg|^2 - \frac{2cz}{r} \times l + \frac{bz}{k} = \frac{ss}{rr} \times \overline{2bz - fh}.$$

From the Resolution of which the Value of z , and from thence the Position of the Point D , will be determined.

Geometrically.

If, in AB , there be taken $AF = a$, and FH be drawn,



to make the Angle $AFH = ACB$ and meet AC in H ; then, the Ratio of FH to AF (by Reason of the similar Triangles AFH , ACB) being the same with That of BC to AC , it is evident that FH is given $= b$.

Moreover, if HG be drawn, making the Angle $AHG = AEC$, it will likewise appear, that both HG and AG are given:

For, since $AG : AH :: AC : AE$,

and $AF : AH :: AC : AB$;

it follows that $AG : AF :: AB : AE$; whence AG is given: And then it will be $a : e (:: AC : EC) :: AG : HG$. Whence HG is given; and, from thence, the following

Construction.

Take $AF = a$, and make the Angle $AFG = AEB$; also take $AI = e$, and draw IK parallel to FG ; moreover, from the Centers F and G , with the Intervals b and AK , describe two Arcs, and from the Point H of their Intersection draw HF and HA , then a Line, BC , drawn to make the Angle $ABC = AHF$, will cut AH , produced, in the Point required.

As to the Trigonometrical Solution, it is too obvious, from the Construction, to need an Explanation. But it

it will be proper to take Notice that the Problem itself becomes impossible, when the two Arcs, described from the Centers F and G, neither cut, nor touch, each other; that is, when the Distance FG is, either, greater than the Sum, or less than the Difference, of b and AK.

PROBLEM LVII.

Three Points, A, B and E, being given, to determine a fourth Point C, so that Lines (AC, BC, EC) drawn from thence to the three former, may have given Differences. (Provided the Difference of no two of the said Lines be given greater than the Distance of the two given Points from whence they are drawn.)

Supposing the given Points to be joined, put $AB = a$,

$AE = b$, $AC = x$,

$BC = x + p$, and

$EC = x + q$ (where

p and q represent the

given Differences).

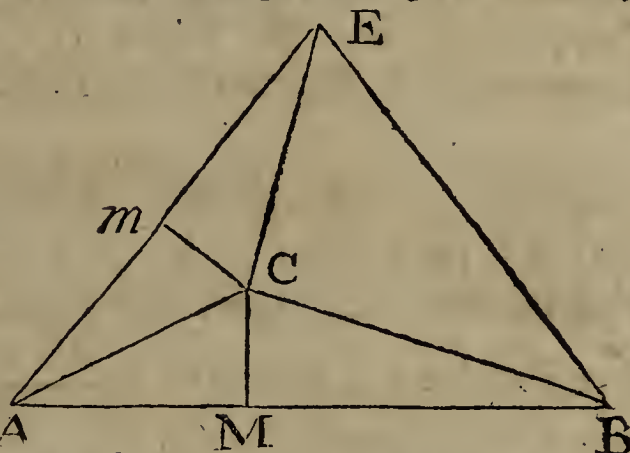
Then, if upon AB

and AE, the Per-

pendiculars CM

and Cm be let fall,

it will be (by a



known Property of Triangles) $AB (a) : BC + AC$

$$(2x + p) : : BC - AC (p) : BM - AM = \frac{2px + pp}{a} :$$

Whence $AM = \frac{1}{2} a - \frac{2px + pp}{2a}$: And, by the very

same Argument, $Am = \frac{1}{2} b - \frac{2qx + qq}{2b}$. Which two

Values, by putting $\frac{1}{2} a - \frac{pp}{2a} = f$, and $\frac{1}{2} b - \frac{qq}{2b} = g$

(for the sake of Brevity) will become $f - \frac{px}{a}$, and $g - \frac{qx}{b}$.

Moreover,

Moreover, if the Sine, and the Co-sine of the given Angle MAm , to the Radius r , be denoted by s and c , respectively, it will appear, from the Problem preceding

the Last, that $\overline{AM}^2 + \overline{Am}^2 - \frac{2c}{r} \times AM \times Am = \frac{ss}{rr} \times$

\overline{AC}^2 ; or, (in Species) $f - \frac{px}{a} + g - \frac{qx}{b} - \frac{2c}{r}$

$\times f - \frac{px}{a} \times g - \frac{qx}{b} = \frac{s^2 x^2}{r^2}$. Which Equation may

be reduced to $\frac{pp}{aa} + \frac{qq}{bb} - \frac{2cpq}{rab} - \frac{ss}{rr} \times x^2 -$

$\frac{fp}{a} + \frac{gq}{b} - \frac{cgp}{ra} - \frac{cfq}{rb} \times 2x = -f^2 - g^2 - \frac{2cfs}{r}$:

Whence, by substituting for the Coefficients of the Powers of x , &c. the Value of x , will be found, and from thence the Position of the Point C.

Geometrically.

If AB be bisected in F, and FG be taken a Third Proportional to $2AB$ and PQ (BC-AC), it is evident, from Problem 49, that $GM = \frac{AC \times PQ}{AB}$.

And, for the very same Reasons, if AE be bisected in f , and fg be taken a Third Proportional to $2AE$ and RS (EC-AC), we shall also have $gm = \frac{AC \times RS}{AE}$.

Whence it appears that GM is to gm , in the given Ratio of $\frac{PQ}{AB}$ to $\frac{RS}{AE}$, or of PQ to $\frac{AB \times RS}{AE}$, or,

lastly, of GI to gi ; by taking $GI = PQ$, and $gi = \frac{AB \times RS}{AE}$.

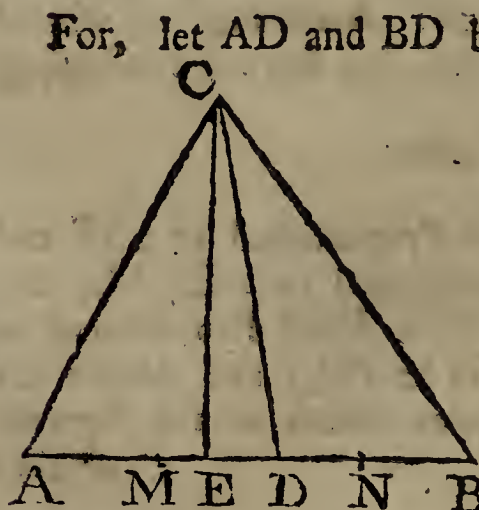
If

diculars GH, gH; IK, iK, as above specified; let an Arch of a Circle, from the Center K, at the Distance of AB be next described, and through A let HN be drawn, meeting it in L; then a Right-line AC drawn parallel to That joining the Points L and K, will cut *de* in the Point required; as is evident from the above-mention'd Problem. By Means of which the Trigonometrical Solution will also be brought out.

In this Construction, the Solution of the Problem for describing a Circle to touch three given Circles, is determined.

L E M M A.

If from the Vertex of any plane Triangle ABC, a Right-line CD be drawn, to divide the Base AB in the Ratio of two given Numbers *n* and *m* (so that $AD : BD :: n : m$); then the Sum of the Multiples of the Squares of the two Sides of the Triangle, whose Factors are the said given Numbers, taken alternately, will be equal to the Sum of the Rectangle of the two Parts of the Base and the Square of the dividing Line, repeated as often as there are Units in the two proposed Numbers (that is, $m \text{ times } \overline{AC}^2 + n \text{ times } \overline{BC}^2 = m + n \text{ times } AD \times BD + \overline{DC}^2$)



For, let AD and BD be bisected in M and N, and upon AB let fall the Perpendicular CE; then will $\overline{AC}^2 - \overline{DC}^2 = AD \times 2ME$ (Elem. 8. 2). Whence it is evident that $m \text{ times } \overline{AC}^2 - m \text{ times } \overline{DC}^2 = m \text{ times } AD \times 2ME$.

And, by the very same Argument, it appears that $n \text{ times } \overline{BC}^2 - n \text{ times } \overline{DC}^2 = n \text{ times } BD \times 2NE$. Therefore,

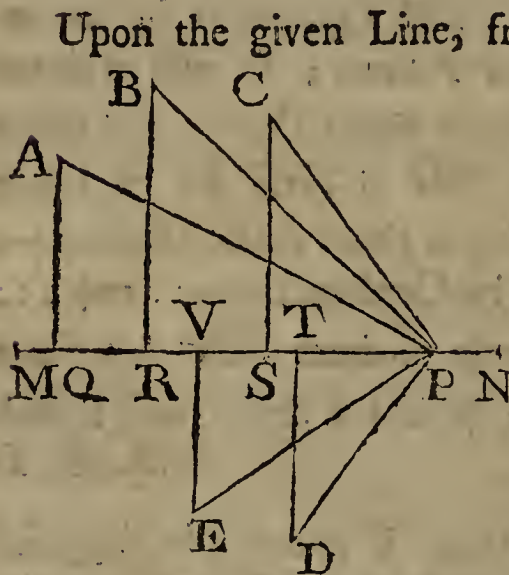
Therefore, by adding these equal Quantities together,
 m times $\overline{AC}^2 + n$ times $\overline{BC}^2 - m + n$ times $\overline{DC}^2 =$
 m times $AD \times 2ME + n$ times $BD \times 2NE$. But $m :$
 $n :: BD : AD$ (by Hyp.) $:: BD \times 2NE : AD \times 2NE$
 (Elem. 1. 4); and therefore n times $BD \times 2NE = m$
 times $AD \times 2NE$.

Let the latter of these equal Quantities be wrote
 above, in the Room of the Former; so will m times
 $\overline{AC}^2 + n$ times $\overline{BC}^2 - m + n$ times $\overline{DC}^2 = m$ times
 $AD \times 2ME + m$ times $AD \times 2NE = m$ times
 $AD \times 2ME + AD \times 2NE = m$ times $AD \times AB = m + n$
 times $AD \times BD$ (because, by Construction $m : m + n ::$
 $BD : AB$); and consequently m times $\overline{AC}^2 + n$ times
 $\overline{BC}^2 = m + n$ times $\overline{DC}^2 + m + n$ times $AD \times BD$
 Q. E. D

It is plain from hence that, if m and n be supposed
 to denote two given Lines, instead of Numbers, the
 Sum of the Solids $m \times \overline{AC}^2$ and $n \times \overline{BC}^2$ will be equal
 to the Sum of the Solids $m + n \times \overline{DC}^2$ and $m + n \times$
 $AD \times BD$.

P R O B L E M LVIII.

From any Number of given Points, A, B, C, D, &c. to draw as many Lines AP, BP, CP, &c. to meet in a Right-line MN given by Position, so that the Sum of all their Squares may be a given Quantity.



Upon the given Line, from all the given Points, let fall Perpendiculars, AQ, BR, CS, &c. which Perpendiculars, as well as the Distances between Them, are all given from the Position of MN: Put, therefore, $AQ = a$, $BR = b$, $CS = c$, $DT = d$, $EV = e$, $QR = r$, $QS = s$, $QT = t$, &c. Also put $QP = x$, and the given Sum of all the proposed Squares $= k^2$.

Then, RP being $= x - r$, SP $= x - s$, &c.

We shall have, by Elem. 7. 2.

$$\left. \begin{aligned} \overline{AP}^2 &= aa + xx \\ \overline{BP}^2 &= bb + xx - 2rx + rr \\ \overline{CP}^2 &= cc + xx - 2sx + ss \\ \overline{DP}^2 &= dd + xx - 2tx + tt \\ &\text{\&c.} \end{aligned} \right\} = k^2$$

Put $aa + bb + cc + dd$ &c. $= \alpha^2$, $r + s + t$ &c. $= \beta$, and $rr + ss + tt$ &c. $= \gamma^2$; and let the Number of the given Points be denoted by n :

Then our Equation will be reduced to

$$\alpha^2 + nx^2 - 2\beta x + \gamma^2 = k^2:$$

From whence x is found $= \frac{\beta}{n} \pm \sqrt{\frac{k^2 - \alpha^2 - \gamma^2}{n} + \frac{\beta^2}{nn}}$.

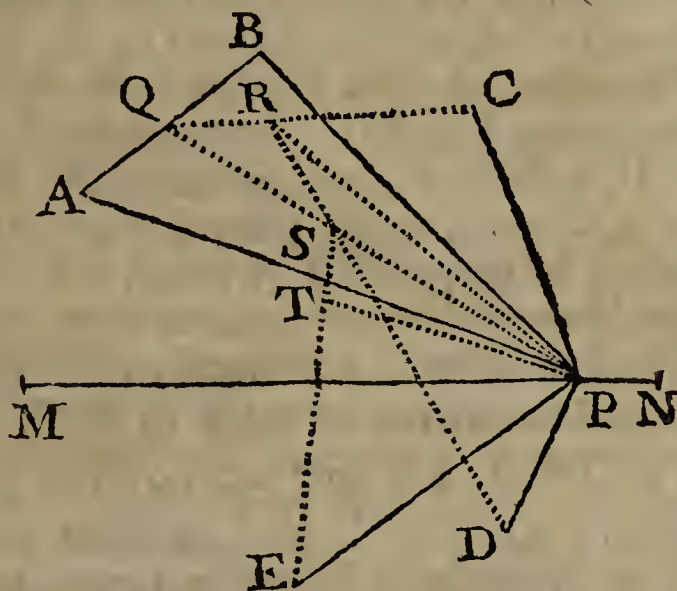
Geometrically.

If PQ be drawn to bisect the Distance AB, then will $\overline{AP}^2 + \overline{BP}^2 = 2AQ \times BQ + 2\overline{QP}^2$, by the Lemma.

And,

And, if QC be drawn, and QR be taken equal to $\frac{1}{3}$ Thereof; then will $2\overline{QP}^2 + \overline{CP}^2 = 3QR \times CR + 3\overline{RP}^2$, by the Same. Whence, by adding these equal Quantities together, and taking $2\overline{QP}^2$ (common) away, we have $\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 = 2AQ \times BQ + 3QR \times CR + 3\overline{RP}^2$.

Again, by drawing RD, and taking RS = $\frac{1}{4}$ Thereof we have $3\overline{RP}^2 + \overline{DP}^2 = 4RS \times DS + 4\overline{SP}^2$, by the Lemma.



And therefore $\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 + \overline{DP}^2 = 2AB \times BQ + 3QR \times CR + 4RS \times DS + 4\overline{SP}^2$, by the Addition of equal Quantities.

In the same manner, if SE be drawn, and ST be taken = $\frac{1}{5}$ Thereof, we shall have $4\overline{SP}^2 + \overline{EP}^2 = 5ST \times ET + 5\overline{TP}^2$. Whence, again, by the Addition of equal Quantities, $\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 + \overline{DP}^2 + \overline{EP}^2 = 2AQ \times BQ + 3QR \times CR + 4RS \times DS + 5ST \times ET + 5\overline{TP}^2$.

Hence it is evident (without proceeding further) that, let the Number of the given Points be what it will, the Square of the last of the Lines QP, RP, SP, TP, &c. drawn as above, will always be a given Quantity; because the Sum of all the Rectangles $2AQ \times BQ$, $3QR \times CR$, $4RS \times DS$, &c. (or That of their Equals $AB \times QB$, $QC \times RC$, $RD \times SD$, &c.) is given, by the Position of the Points A, B, C, D, &c.

In order, therefore, to the Geometrical Construction let a Rectangle be constituted (*by Elem. 7. 6.*) equal to the Excess of the given Sum of the Squares above the Sum of the said Rectangles: Then find a Mean Proportional between the Length of the Rectangle, so determined, and that Part of its Breadth defined by the Number of the given Points; with which, as a Radius, from the last of the Points Q, R, S, T, &c. let the Circumference of a Circle be described; which will intersect the Line MN, in the Point P, required.

If, instead of the Right-line MN, the Circumference of a Circle (or any other Curve-line) given by Position, be propounded (wherein the Lines, drawn from the given Points, are required to meet), the Method of Construction will, it is manifest, be exactly the same.

And it may not be amiss to observe farther, that if $a \times \overline{AP}^2 + b \times \overline{BP}^2 + c \times \overline{CP}^2 + d \times \overline{DP}^2$, &c. (where a, b, c, d , &c. represent known Quantities, either Numbers or Lines) be supposed given, the Locus of the Point P will, *still*, be the Circumference of a Circle, determined after the same manner, from the premised Lemma.

For, by making in this Case $\left\{ \begin{array}{l} AQ : BQ :: b : a \\ QR : RC :: c : a+b \\ RS : DS :: d : a+b+c \\ ST : ET :: e : a+b+c+d, \end{array} \right.$

and then proceeding as above, it will appear that $a \times \overline{AP}^2 + b \times \overline{BP}^2 + c \times \overline{CP}^2 + \text{&c.}$ is equal to $\overline{a+b} \times AQ \times QB + \overline{a+b+c} \times QR \times RC + \overline{a+b+c+d} \times RS \times SD + \overline{a+b+c+d+e} \times ST \times TE + \overline{a+b+c+d+e} \times \overline{TP}^2$: Whence it is evident that TP is a given Quantity.



P A R T III.

W H E R E I N

The THEORY of GUNNERY, or the
MOTION of PROJECTILES; is considered.

IT is, usually, taken for granted, by *Those* who treat of the Motion of Projectiles, that the Force of Gravity near the Earth's Surface is every-where the same, and acts in parallel Directions; and that the Effect of the Air's Resistance upon very heavy Bodies, such as Bombs and Cannon Balls, is too small to be taken into Consideration.

That the Error arising from the Supposition of Gravity acting uniformly, and in parallel Lines, must be exceeding small, is very obvious; because, even, the greatest Distance of a Projectile above the Surface of the Earth, is inconsiderable in Comparison of its Distance from the Center, to which the Gravitation tends: But then, on the other hand, it is very certain, that the Resistance of the Air, to very swift Motions, is much greater than it has been commonly represented.

Nevertheless, if the Amplitude of the Projection, answering to one given Elevation, be first found by Experiment (which our Method supposes to be done) the Amplitudes in all other Cases, where the Elevations and Velocities do not very much differ from the First, may be determined, to a sufficient Degree of Exactness, from the foregoing Hypotheses: Because, in all such Cases, the Effects of the Resistance will be nearly as the Amplitudes *themselves*; and, were They *accurately* so, the Proportions of the

N 2

Amplitudes,

Amplitudes, at different Elevations, would then be the very same as *in vacuo*.

For this Reason, and to avoid having Recourse to Principles and Calculations no-ways adequate to the Experience and Understanding of Beginners, for whose Use this little Tract is chiefly intended, I shall, in what follows, conform to the Method of other Writers, so far, as to take no Notice of the Air's Resistance; but consider the Motions as performed *in vacuo*.

Now, in order to form a clear Idea of the Subject here proposed, the Path of every Projectile is to be considered as depending on two different Forces; that is to say, on the impellent Force, whereby the Motion is first began (and would be continued, in a Right-line), and on the Force of Gravity, by which the Projectile, during the whole Time of its Flight, is continually urged downwards, and made to deviate more and more from its first Direction.

As whatever relates to the Track and Flight of a Ball (neglecting the Resistance of the Air) is to be determined from the Action of these two Forces, it will be proper, before we proceed to consider their joint Effect, to premise something concerning the Nature of the Motion produced by Each, when supposed to act alone, independent of the Other; to which End the two first, of the four following *Lemmas*, are premised.

LEMMA

L E M M A I.

Every Body, after the impress'd Force, whereby it is put in Motion, ceases to act, continues to move uniformly in a Right-line; unless it be interrupted by some other Force or Impediment.

This is a Law of Nature, and has its Demonstration from Experience and Matter-of-Fact.

COROLLARY.

It follows from hence that a Ball, after leaving the Mouth of the Piece, would continue to move along the Line of its first Direction, and describe Spaces therein proportional to the Times of their Description, were it not for the Action of Gravity; whereby the Direction is changed, and the Motion interrupted.

L E M M A II.

The Motion, or Velocity, acquired by a Ball, in freely descending from Rest, by the Force of an uniform Gravity, is as the Time of the Descent; and the Space fallen thro', as the Square of that Time.

The first Part of the *Lemma* is extremely obvious: For, since every Motion is proportional to the Force whereby it is generated, That generated by the Force of an uniform Gravity must be as the Time of the Descent; because the *whole Effort* of such a Force is proportional to the Time of its Action; that is, as the Time of the Descent.

To demonstrate that the Distances descended are proportional to the Squares of the Times, let the Time of falling thro' any proposed Distance AB be represented by the Right-line PQ; which conceive to be divided into an indefinite Number of very small, equal, Particles, represented, each, by the *Symbol* m ; and let the Distance descended in the First of them be Ac ; in the Second cd ; in the Third de ; and so on.

Then the Velocity acquired being always as the Time from the Beginning of the Descent, it will at the Middle of the first of the said Particles be represented by $\frac{1}{2}m$; at the Middle of the Second, by $1\frac{1}{2}m$; at the Middle of the Third, by $2\frac{1}{2}m$, &c. Which Values constitute the Series $\frac{m}{2}, \frac{3m}{2}, \frac{5m}{2}, \frac{7m}{2}, \frac{9m}{2}, \&c.$

But, since the Velocity, at the Middle of any One of the said Particles of Time, is an exact Mean between the Velocities at the two Extremes thereof, the corresponding Particle of the Distance AB may be therefore considered as described with that mean Velocity: And so, the Spaces $Ac, cd, de, ef, \&c.$ being, respectively, equal to the abovementioned Quantities $\frac{m}{2}, \frac{3m}{2}, \frac{5m}{2},$

$\frac{7m}{2}, \&c.$ it follows, by the continual Addition of

These, that the Spaces $Ac, Ad, Ae, Af, \&c.$ fallen thro' from the Beginning, will be expressed by $\frac{m}{2}, \frac{4m}{2}, \frac{9m}{2}, \frac{16m}{2}, \frac{25m}{2}, \&c.$ Which are, evidently, to one another in proportion, as 1, 4, 9, 16, 25, &c. that is, as the Squares of the Times. Q. E. D.

COROL.

COROLLARY.

Seeing the Velocity acquired in any Number (n) of the aforefaid, equal, Particles of Time (measured by the Space that would be described in one single Particle) is represented by n times m , or $n m$, it will therefore be, as 1 Particle of Time, is to n such Particles, so is $n m$, the said Distance answering to the former Time, to the Distance, $n^2 m$, corresponding to the Latter, with the same Celerity, acquired at the End of the said n Particles. Whence it appears that the Space $\frac{n^2 m}{2}$ (*found above*) thro' which the Ball falls, in any given Time n , is just the Half of That ($n^2 m$) which might be uniformly described with the last, or greatest, Celerity, in the same Time.

SCHOLIUM.

It is found, by Experiment, that any heavy Body, near the Earth's Surface (where the Force of Gravity may be consider'd as uniform) descends about 16 Feet, from Rest, in the first Second of Time.

Therefore, as the Distances fallen thro' are proved above to be, in Proportion, as the Squares of the Times; it follows that, as the Square of 1 Second, is to the Square of any given Number of Seconds, so is 16 Feet, to the Number of Feet a heavy Body will freely descend in the said given Number of Seconds. Whence the Number of Feet descended in any given Time will be found, by multiplying the Square of the Number of Seconds by 16.

Thus the Distance descended in 2, 3, 4, 5, &c. Seconds, will appear to be 64, 144, 256, 400 F, &c. respectively.

Moreover, from hence, the Time of the Descent thro' any given Distance will be obtained, by dividing the said Distance, in Feet, by 16, and extracting the square Root of the Quotient; or, which comes to the

N 4

same

same Thing, by extracting the square Root of the whole Distance, and then taking $\frac{1}{4}$ of that Root for the Number of Seconds required. Thus, if the Distance be supposed 2640 Feet; then, by either of the two Ways, the Time of the Descent will come out 12, 84 Seconds, or 12" : 50".

It appears also (*from the Corol.*) that the Velocity per Second (in Feet), at the End of the Fall, will be determined, by multiplying the Number of Seconds in the Fall by 32: Thus it is found that a Ball, at the End of 10 Seconds, has acquired with a Velocity of 320 Feet per Second.—After the same Manner, by having any two of the four following Quantities, *viz.* the Force, the Time, the Velocity, and Distance, the other Two may be determined: But this not being absolutely necessary in what follows (though equally useful in other Disquisitions) I shall put down the several Rules, or Equations below, in a Note *, to be taken, or omitted, at Pleasure.

LEMMA

* Let the Space freely descended by a Ball, in the first Second of Time (which is as the accelerating Force) be denoted by f ; also let t denote the Number of Seconds wherein any Distance, d , is descended; and let v be the Velocity, per Second, at the End of the Descent: Then will

$$v = 2ft = 2\sqrt{fd} = \frac{2d}{t}$$

$$t = \sqrt{\frac{d}{f}} = \frac{v}{2f} = \frac{2d}{v}$$

$$d = ftt = \frac{vv}{4f} = \frac{tv}{2}$$

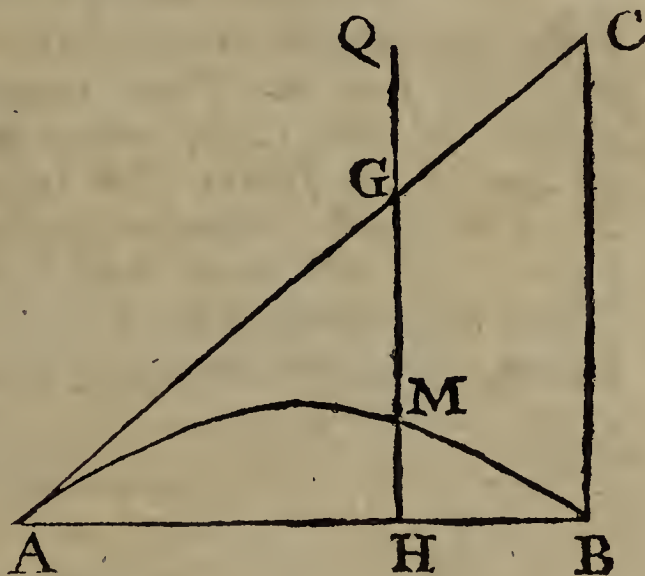
$$f = \frac{d}{tt} = \frac{v}{2t} = \frac{vv}{4d}.$$

All which Equations are very easily deduced from the two original Ones, $d = ftt$, and $v = 2ft$, already demonstrated; the Former in the Proposition itself; and the Latter, in the Corollary to it; by which it appears that the
Measure

L E M M A III.

A Ball, projected in the Direction of a Right-line AC making an Angle with the Horizon AB, is, by Means of its Gravity, deflected continually, more and more, from its first Direction; but the Celerity with which it approaches any Perpendicular (BC) to the Horizon, is neither increased nor decreased by the Action of Gravity.

For, let a Line HMQ , perpendicular to the Horizon, be conceived to move parallel to itself, towards BC , along with the Ball: Then, as the Gravity always acts in this Line, it can have no Effect in making it, either, move faster or slower towards BC ; but is wholly employ'd in drawing down the Projectile along the same, from its first Direction AC ; and thereby causing it to describe a Curve-line AMB .



COROL.

Measure of the Velocity at the End of the first Second is $2f$; whence the Velocity (v) at the End of t Seconds must consequently be expressed by $2f \times t$ or $2ft$.

Having proceeded thus far, I shall here take the Opportunity to point out, and rectify an inadvertent Expression, at p. 230 of my Book of Fluxions, relating to this Subject. — It is there said, by way of Remark, that, whatever Ratio the Times have with Respect to the Distances descended, &c. the same also will the Velocities have, being

COROLLARY.

Hence it appears that the Projectile, at the End of a given Time, is in the very same vertical Line HG, as it would be in, if Gravity was not to act; and that the horizontal Distance AH, as well as the Hypothenufe AG, is proportional to the Time wherein the Projectile actually moves through the Arch AM, corresponding to the said Distance.

being proportional to the Times. *Which Observation, it is clear from the Words themselves, ought to be restrained to those Cases, where the Velocities are proportional to the Times; that is, where the accelerating Forces are equal. Therefore, instead of the said Observation, as it now stands, read, What is above demonstrated with respect to the Times, holds also in the Velocities, when the accelerating Forces are equal.*

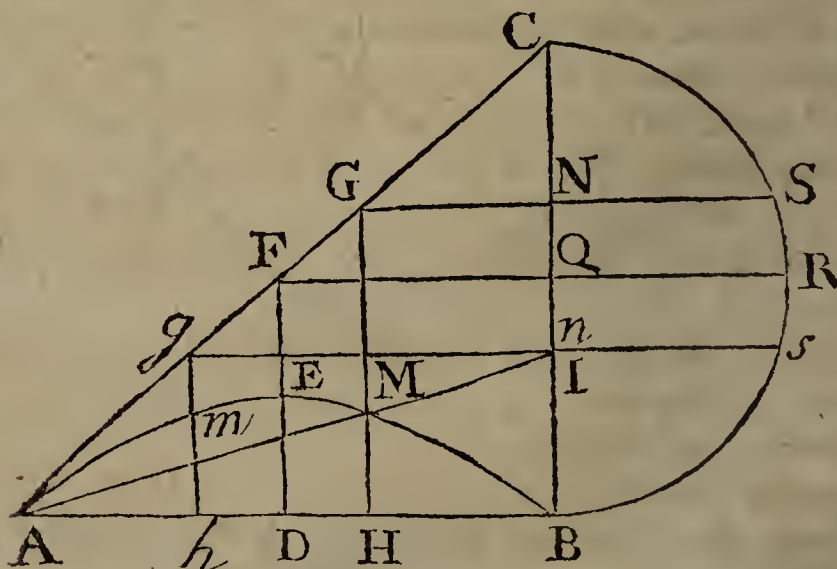
LEMMA

from Rest, are as the Squares of the Times; therefore their Equals GM and CB are likewise as the Squares of the same Times; that is, of the Times wherein AM and AMB are described: Which are to each other as \overline{AG}^2 to \overline{AC}^2 (by Corol. to Lem. 3.)

COROL. I.

If B be taken as the Point where the Projectile impinges on the Horizon AB, and in BC (perpendicular to AB) there be taken $CI = GH$; then a Right-line, drawn from I to A, will cut the vertical Line HG in the very Point (M) thro' which the Center of the Projectile passeth.

For $\overline{AC}^2 : \overline{AG}^2 :: BC : MG$, *as above.*
 and $\overline{AC}^2 : \overline{AG}^2 :: \overline{BC}^2 : \overline{HG}^2$, *by sim. Triangles.*
 Theref. $BC : MG :: \overline{BC}^2 : \overline{HG}^2$, *by Equality of Ratios:*
 and consequently $MG \times BC = \overline{HG}^2$.



This, turn'd into an Analogy, gives $BC : HG :: HG : MG$; and so likewise is BC to CI (because of the parallel Lines). Whence it is evident that HG and CI are equal to each other, in all Positions of HG : By Means whereof, as many Points in the Curve, as you please, may be determined.

COROL.

COROL. II.

If GNS be drawn, parallel to AB, intersecting BC in N, and meeting the Circumference of a Semi-circle, described upon BC, in S; it will further appear, that the Height (HM) of the Ball is everywhere a Third-proportional to BC and NS. For, since $BC : HG :: HG : MG$, it follows, by Division, that, $BC : CN :: HG (BN) : HM = \frac{CN \times BN}{BC} =$

$\frac{NS^2}{BC}$ (*Elem.* 16. 3). Hence it is evident that the Projectile will be at its greatest Height, when it has performed, just, the Half of its Flight; since it is well known that

NS (and consequently $\frac{NS^2}{BC}$) is the greatest possible when it coincides with the Radius QR.

It appears moreover that the Heights *hm*, HM, of the Ball, in any two vertical Lines, equally distant from That passing thro' the highest Point E of the Trajectory, are equal; because the corresponding Ordinates *ns* and NS are equal, as being equally distant from the Center Q of the Semi-circle. — Lastly, it is apparent that the

greatest Altitude DE is $= \frac{1}{4} BC$; because $\frac{NS^2}{BC}$, when

NS coincides with QR, becomes $= \frac{\frac{1}{4} BC^2}{BC} = \frac{1}{4} BC$.

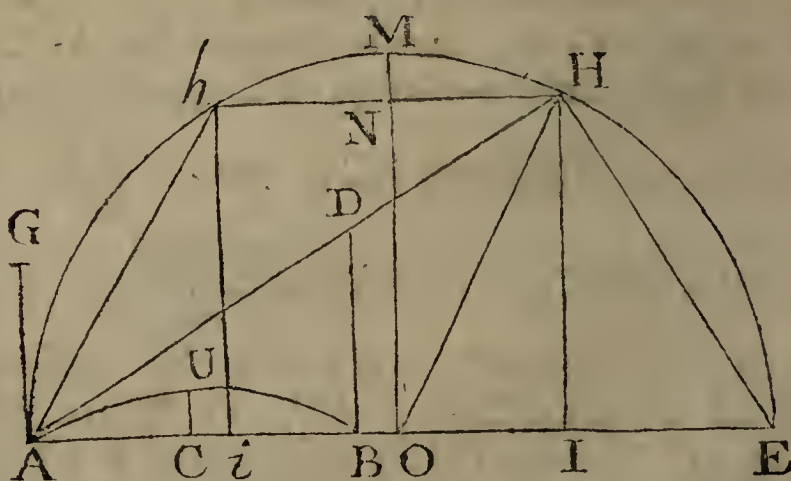
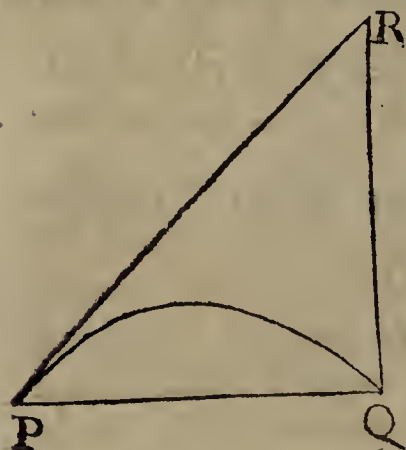
Which may be otherwise made out, by considering, that, as AF is but the Half of AC, its Square will be only $\frac{1}{4}$ of the Square of BC; and therefore FE only $\frac{1}{4}$ of BC (*by the Lem.*). And so, DF being $= \frac{1}{2} BC$ (*by sim. Triang.*) DE, as well as FE, must be $= \frac{1}{4} BC$.

PROPOSITION

PROPOSITION I.

The Amplitude, or horizontal Range of a Piece, with a given Charge of Powder, at an Elevation of 45 Degrees, being known, from Experiment; to determine the Elevation so as to hit an Object, at a given Distance, on the plane of the Horizon; the Quantity of Power remaining the same.

Let PQ be the given Amplitude, at an Elevation (QPR) of 45 Degrees; let AB be the given Distance of the Object, and BAD the required Elevation: In AB, produced, take AO=PQ, with which as a Radius, from the Center O, let a Semi-circle AME be described, and let AD, produced, meet the Periphery thereof in H; join E, H, and make DB and HI perpendicular to AE.



Since the Charge of Powder, or the Velocity at both the Elevations QPR and BAD, is supposed to be the same, the Times of Flight, during which the Distances PR and AD would be uniformly described with that Velocity, will therefore be to each other, directly as the said Distances; and consequently $\overline{PR}^2 : \overline{AD}^2 :: RQ$

RQ : DB (by Principles already explained. *vid.* Lem. 3 and 4).

But $\overline{PR}^2 (= \overline{PQ}^2 + \overline{RQ}^2) = 2\overline{RQ}^2$ (because the Angle P ($= 45^\circ$) = R). Hence the above Proportion becomes $2\overline{RQ}^2 : \overline{AD}^2 :: RQ : DB$; from which we have $\overline{AD}^2 = 2RQ \times DB = AE \times DB$, by Construction.

Moreover, from the similar Triangles ABD and AHE, we have $AE : EH :: AD : DB$: Whence $EH \times AD = AE \times DB = \overline{AD}^2$ (*p. above*) and consequently $EH = AD$: And so, the Triangles EHI and ADB being equiangular, it is plain that HI is also equal to AB. Whence follows this easy

Construction.

With an Interval equal to the given Amplitude, at the Elevation of 45° , let a Semi-circle AME be described; make OM perpendicular to the Diameter thereof, in which take ON equal to the given Distance AB, and thro' N, parallel to AE, draw HN*b*, intersecting the Circumference in H and *b*; then either of the Directions AH, or A*b*, will answer the Conditions of the Problem.

Q. E. I.

COROL. I.

If OH be drawn, the Angle EOH will be $= 2EAH$ (*Elem.* 9. 3.) and it will be, as OH (PQ) : HI (AB) :: Radius : Sine EOH; that is, in Words, as the given Amplitude, at 45° Elevation, is to any other proposed Amplitude, so is the Radius, to the Sine of twice the Elevation corresponding to the Latter. From whence it is evident, that the horizontal Amplitudes, at different Elevations, are to one another as the Sines of the Doubles of the said Elevations; and that, the Amplitude of the Projection at an Elevation of 45 Degrees (when HI coincides with MO) is the greatest possible.

COROL.

COROL. II.

Since it is found above that $AE \times DB (= \overline{AD}^2) = \overline{EH}^2$, it follows, that $\overline{AE}^2 : \overline{EH}^2 :: \overline{AE}^2 : AE \times DB :: AE : DB :: \frac{1}{2} AO (\frac{1}{4} AE) : CV (\frac{1}{4} DB. \textit{vid. Corol. 2. to Lem. 4.})$ that is, as the Square of the *Radius*, is to the Square of the *Sine* of the Angle of Elevation, so is Half the greatest Horizontal Amplitude, to the greatest Altitude of the Projectile.

Hence it appears, that the Distance which the Ball would ascend, if projected in a vertical Direction (*usually call'd the Impetus*) is just one Half of the greatest Amplitude; since, in this Case, the *Sine* of the Elevation becomes equal to the *Radius*.

Therefore, as a Body (*in vacuo*) ascends and descends with the same Celerity; and seeing the Distance AG, expressing the perpendicular Ascent, is as the Square of the Celerity at A, (*p. Lem. 2*); it follows that the greatest horizontal Amplitude AO, being $= 2AG$, is also as the Square of the same Celerity.

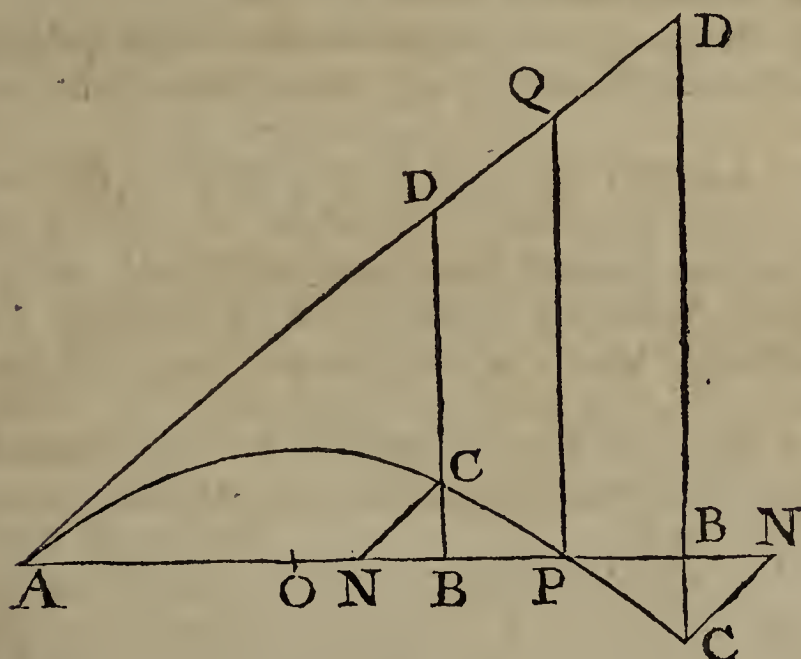
From whence, and *Corol. I*, it is manifest, that the Amplitudes, when both the Elevations, and the Velocities, differ, will be to each other in a *Ratio* compounded of the *Ratio* of the *Sines* of the Double Elevations, and the *Duplicate Ratio* of the Velocities.

PROPOSITION

PROPOSITION II.

The greatest horizontal Amplitude, and the Angle of Elevation being given; to find at what Distance the Piece ought to be planted, to hit an Object, whose Height above, or Depression below, the Level of the Piece is also given.

Let BC be the perpendicular Height, or Depression of the proposed Object, and AB the required Distance; let BC, produced, meet the Line of Direction AD in D, and let P be the Place where the Path of the Projectile (produced) meets the Level of the Piece: Make PQ perpendicular to AP, and CN parallel to AD.



By the last Proposition it will be, as *Radius* : *Sin. 2A* :: the greatest horizontal Amplitude, to the Distance AP; which, therefore, is known.

Moreover, it appears from the fourth Proportion, in *Corol. 1, to Lem. 4,* that $PQ : BD :: BD : CD$.

But $\left\{ \begin{array}{l} PQ : BD :: AP : AB \\ BD : CD :: AB : AN \end{array} \right\}$ by similar Triangles.

Therefore, by Equality of Ratios,
 $AP : AB :: AB : AN$;

O

Whence

duced, in E; join E, H, and let HI, AN, and QR, be perpendicular to AE, AO, and PQ respectively, and let BC be produced to meet AH in D.

It appears, from what has been already delivered, that $\overline{AD}^2 : \overline{PR}^2 :: DC : RQ$. Whence, \overline{PR}^2 being $= 2\overline{PQ}^2 = 2\overline{AG}^2 = \frac{1}{2} \overline{AE}^2$, and $RQ = PQ = \frac{1}{2} AE$ (by Construction) we have $\overline{AD}^2 : \frac{1}{2} \overline{AE}^2 :: DC : \frac{1}{2} AE$; and consequently $\overline{AD}^2 = AE \times DC$.

But, the Triangles ADC and AEH being equiangular (because $ADC = DAN = E$, and DAC common) we likewise have $AD : DC :: AE : EH$; and therefore $AE \times DC = AD \times EH = \overline{AD}^2$ (p. above) whence $EH = AD$; and so, the Triangles ADB and EHI being equiangular, we likewise have $HI = AB$; and from thence this easy

Construction.

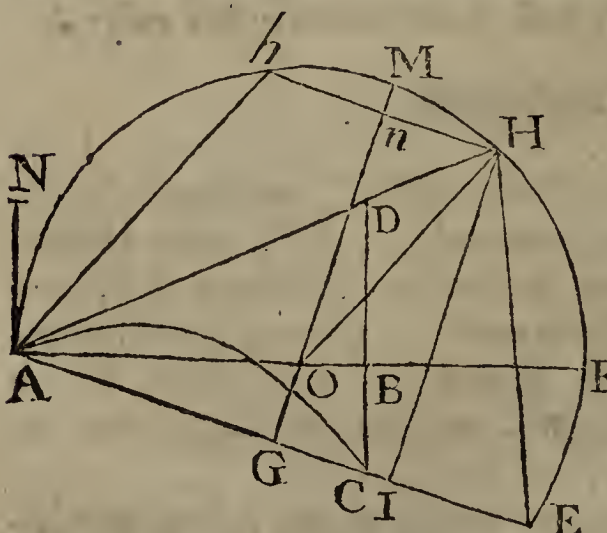
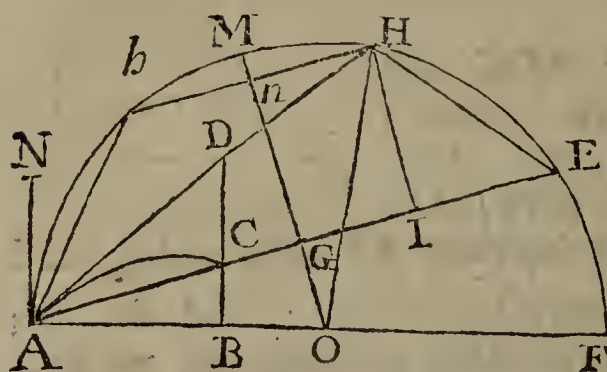
Having described the Circle AHEF, as above directed, and drawn MG perpendicular to AE, take therein Gn equal to the given horizontal Distance AB, and thro' n, parallel to AE, draw Hh, cutting the Circumference of the Circle in H and h; then either of the Directions AH, or Ah, will answer the Conditions of the Problem.

From this Construction we have the following Calculation, viz. As $AB : BC :: AG (PQ) : GO$; which, added to, or subtracted from Gn (AB), gives On.

Then, as $AG (PQ) : On :: Co-sine \text{ of } OAG : Co-sine \text{ } HOn = (HAh)$ the Difference of the two required Elevations: From which the Elevations *themselves* will be known: And from whence, if BC be supposed to vanish, the Solution to *Prop. I.* will be obtained; being only a particular Case of That above. Q. E. I.

COROL. I.

Hence the Time of the Flight may likewise be determined: For, $S. AHE (ACD) : S. HAE :: AE : (\frac{S. HAE}{S. AHE} \times AE =) HE = AD$ (*p. above*) Which being proportional to the Time of the Flight (*vid. Lem. 3.*) it follows that the said Time will, always, be as $\frac{S. HAE}{S. ACD}$; because AE is supposed constant. There-



fore, if the Time of the perpendicular Ascent, or Descent, thro' the Impetus AN , be found, and denoted by T , it is evident that the Time sought will be truly represented by $\frac{S. HAE}{S. ACD} \times 2T$; that is, the *Co-sine* of the Object's Elevation above, or Depression below the Level of the Piece, will be to the *Sine* of the Elevation of the Piece above the Object, as twice the Time of the perpendicular Ascent, or Descent, is to the true Time of the Flight.

COROL.

COROL. II.

If the Elevation of the Piece, together with the Distance, and the Height (or Depression) of the Object, be given, the *Impetus* may, from hence, be also found.

For, first, it will be $AB : BC :: \text{Radius} : \text{Tang. BAC}$; whence (as BAD is given) EAH will likewise be known.

Then, $S. \text{EAH (CAD)} : S. \text{AHE (ACD)} :: \text{HE (AD)} : \text{AE}$. Also $S. \text{ADC} : \text{Radius} :: AB : AD$.

Therefore, by compounding these Proportions, we have $S. \text{CAD} \times S. \text{ADC} : \text{Rad.} \times S. \text{ACD} :: AB : AE$; which is 4 times the required *Impetus* (*vid. Corol. 3. to Prop. 1*).

COROL. III.

Moreover, if the Elevation, and the *Impetus* be given, the Amplitude of the Projection on an ascending, or descending Plane ACE, whose Inclination is given, may from hence be derived.

For, $S. \text{AHE (ACD)} : S. \text{EAH (CAD)} :: AE : \text{EH (AD)}$. And $S. \text{ACD} : S. \text{ADC} :: AD : AC$.

By the Composition of which Proportions we have $\square S. \text{ACD} : S. \text{CAD} \times S. \text{ADC} :: AE : AC$; whence AC is given.

COROL. IV.

Hence, also, may the Ratio of the Amplitudes, on the same Plane, at different Elevations, be deduced: For the first and third Terms, of the last Proportion, continuing invariable, the Ratio of the 2^d and 4th. will likewise be invariable; that is, the Rectangle under the *Sine* of the Elevation above the Plane, and the *Cosine* of the Elevation above the Horizon, in any one Case, will be to the like Rectangle, in any Other, as the Amplitude in the former Case, to the Amplitude in the Latter.

COROL. V.

But, if the Elevation be supposed constant, and the Plane's Inclination to vary; then, since, by the above Proportion, AC is *universally*
$$= \frac{S. CAD \times S. ADC}{\square Sine ACD} \times$$

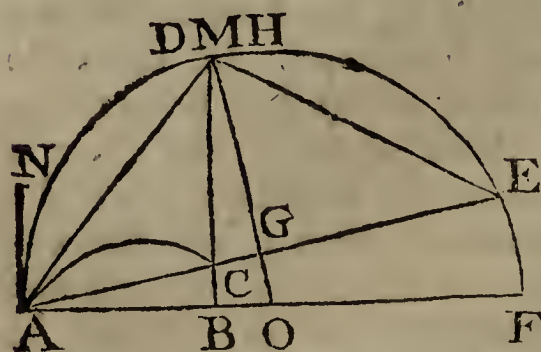
AE (where AE and the Angle ADC are supposed constant) it follows that the Amplitude will be, barely, as

$\frac{S. CAD}{\square S. ACD}$; that is, *inversely*, as the square of the *Co-*

sine of the Inclination of the Plane, apply'd to the *Sine* of the Elevation above the Plane.—If both the Inclinations, and the Elevations, differ, it will appear, from the same Equation, that, the Amplitude will be, *universally*, as the Rectangle of the *Sine* of the Elevation above the Plane, and the *Co-sine* of the Elevation above the Horizon, apply'd to the Square of the *Co-sine* of the Plane's Inclination.

COROL. VI.

Since it is proved that HI is, always, equal to AB, it is evident that, when the Former coincides with MG, and thereby becomes a *Maximum*, the Latter will also be a *Maximum*: In which Circumstance AC will likewise be a *Maximum*; and the Point D will then coincide with M and H (*as in the annexed Figures*) because AD and EH are always equal to each other.



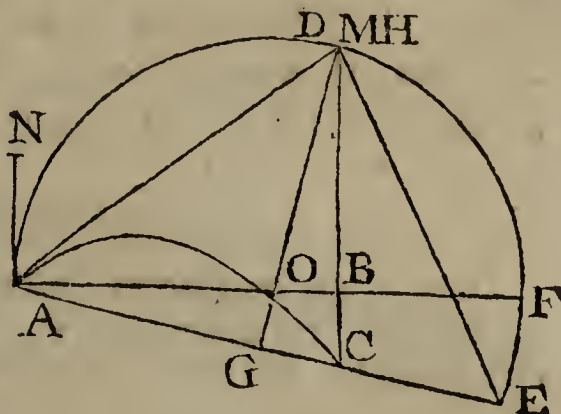
bisects the Angle EAN, included between the Plane and the Zenith.

Therefore, since, in this Case, the Angle HAE is $(= E) = NAH$, it appears that the Amplitude, on any inclin'd Plane, will be the greatest possible when the Line of Direction AH

COROL.

COROL. VII.

Hence the greatest Amplitude, on any inclin'd Plane, with a given Impetus, may be determined: For the right-angled Triangles AOG and HOB, having $AO = HO$, and the Angle O, common, are equal in all respects: Therefore it will be, as the Tangent of AHG (or BAH, the Piece's Elevation) is to the Tang. of CHG (or CAB, the Plane's Inclination) so is AG (twice the given Impetus) to the Difference CG between the said Double Impetus and the Amplitude sought.



COROL. VIII.

Hence, also, if the greatest Amplitude, on an inclin'd Plane be given, the Impetus may be found: For, it will be as the Radius, is to the Sine of the Plane's Inclination BAC, so is the given Amplitude AC to the Difference (BC or CG) betwixt it and twice the Impetus. *Vid. Corol. 2. to Prop. I.*

COROL. IX.

But if, instead of the Plane's Inclination, the perpendicular Height, or Depression of the Object above, or below the Level of the Piece, be given; then, AC being $= AG \mp BC$, and $\overline{AB}^2 (= \overline{AC}^2 - \overline{BC}^2) = \overline{AG}^2 \mp 2AG \times BC$, the greatest Distance AB, at which the Ball can possibly hit the Object, will therefore be $= \sqrt{AG \times AG \mp 2BC}$. From which, as all the Sides of the Triangle ABC are given, the Angle

BAC will likewise be known ; and, from thence, the Elevation, BAH, by *Corol. 6.*

From the latter of the two Cases here consider'd (where the Object is supposed below the Level of the Piece, as in *Fig. 2.*) the greatest Amplitude of a Ball, projected from a given Height above the Plane of the Horizon, is given ; being $= \sqrt{AG \times AG + 2BC}$, (as above) in which BC denotes the given Height.

COROL. X.

But, if the horizontal Distance AB is given, and it be required to determine the greatest Height the Ball can possibly reach, in the Perpendicular BCD ; we shall then have, $HG (AB) : AG :: Radius : Tang.$ of the Elevation (BAH or AHG) ; and, as $Radius : Tang. BAC (= 2BAH \approx 90) :: AB : BC$; which therefore is known. But, because $AC = AG + BC$, and $BC^2 = AC^2 - AG^2$, the Value of BC will, also, be truly expressed by $\frac{AG^2 \approx AB^2}{2AG}$.

COROL. XI.

Lastly, if there be given the perpendicular Height, or Depression, of the Object, and its horizontal Distance, in order to determine the Elevation, and the least Impetus, to hit the Object : Then it will be as $AB : BC :: Radius : Tang. BAC$; whence the Elevation BAH is also known, by *Corol. 6* : And, as $Radius : Tang. AHG (BAH) :: HG (AB) : AG$; the Half of which is the Impetus, by *Prop. 1. Corol. 2.*

Here

Here follow the practical Solutions of the several Cases depending on the foregoing THEORY.

I. Of Projections made on the Plane of the Horizon.

P R O B L E M I.

The greatest Amplitude of a Piece being known (from Experiment) to find the Amplitude, at any proposed Elevation.

SOLUTION.

As the Radius, is to the Sine of double the proposed Elevation, so is the given, to the required, Amplitude (*by Prop. I. Cor. I.*)

Ex. Let the gr. Amp. be 8000 Feet, and the given Elev. $30^{\circ} : 16'$.

Then, as Radius	—————	10. 0000
to Sine $60^{\circ} : 32'$	—————	9. 9398
so is	———— 8000 ————	3. 9030
Amp. req. 6965 F.	—————	3. 8428

P R O B L E M II.

The Impetus, or the greatest Amplitude (which is the Double thereof) being known, to find the Elevation, to strike an Object at a given Distance.

SOLUTION.

As the greatest Amplitude, is to the given Distance; so is the Radius, to the Sine of the Double Elevation (*by Prop. I. Corol. I.*)

Ex. Let the gr. Amp. be 7500 F. and the given Distance 5620 F.

Then,

Then, as 7500 ————— 3. 8750
 is to 5620 ————— 3. 7497
 so is Radius ————— 10. 0000

to the sine of $48^\circ : 32'$, or $131^\circ : 28'$ — 9. 8747

Therefore $24^\circ : 16'$ is the lower, and $65^\circ : 44'$ the higher Elevation, required.

P R O B L E M III.

The Angle of Elevation, and the Distance of an Object, being given, to find the Impetus, so as to strike the Object.

SOLUTION.

As the Sine of twice the Elevation; is to the Radius, so is the Distance of the Object to twice the Impetus (by Prop. I).

Ex. Let the Elev. be $32^\circ : 12'$ and the given Dist. 6500. F.

Then, as Sine $64^\circ : 24'$ ————— 9. 9551
 is to Radius ————— 10. 0000
 so is — 6500 ————— 3. 8129
 to — — 7208 ————— 3. 8578

Whence 3604 is the Impetus required.

P R O B L E M IV.

The Amplitude, at any one known Elevation being given, to find the Amplitude at any other known Elevation.

SOLUTION.

As the Sine of Double the first Elevation, is to the Sine of Double the Second, so is the Amplitude at the Former, to that at the Latter, (by Prop. I).

Ex. Let the first Elevⁿ. be $25^\circ : 12'$, the 2^d. $36^\circ : 15'$, and the given Amp. 5250. F.

Then,

Then, as Sine $50^{\circ} : 24' -$ Co. Ar. 0. 1132
 to Sine $- 72^{\circ} : 30' -$ 9. 9794
 so is $- 5250 \text{ F.} -$ 3. 7201
 to the Amp. req. 6498 F. $- 113. 8127$

P R O B L E M V.

The Amplitude of the Projection, with a given Quantity of Powder, being known; to find the Requisite of Powder, so as to strike an Object at a given Distance; the Elevation remaining the same.

SOLUTION.

As the given Amplitude, is to the proposed Distance, so is the given Weight, or Quantity of Powder, to the Quantity sought, nearly *.

Ex. Suppose the Requisite of Powder to throw a Shot 4000 Feet, at 45° Elevation, to be 16 lb. What Quantity is necessary, to strike an Object at the Distance of 5000 Feet.

Here it will be, as $4000 : 5000 :: 16 : 20 \text{ lb.}$ the Quantity sought.

* In this Solution, the Velocity communicated to the Ball, is supposed to be in the Subduplicate Ratio of the Quantity of Powder; which is not strictly true, especially in large Charges; since a considerable Part of the Powder, in such Cases, is blown out, unfired: There are, besides, other Reasons to be assigned, why, the Velocity cannot be exactly in the Proportion above specified.

P R O B L E M VI.

The Distance of the Object, and the Elevation of the Piece, being given; to determine the Time of the Flight.

SOLUTION.

As the Radius is to the Tangent of the Elevation, so is the given Distance of the Object, *in Feet*, to the Square of 4 times the Number of *Seconds*, required (*vid. p. 184 and 187*).

Ex. Let the Elevation be 32° , and the Distance of the Object 5280 Feet.

Then, Radius	—————	10. 0000
Tang. 32°	—————	9. 7958
Dist. 5280	—————	3. 7226
		<u>13. 5184</u>
57. 44	—————	1. 7592

$\frac{1}{4}$ of which, or $14^s, 36$, is the Time required.

But, when the Elevation is 45 Degrees (which is commonly the Case, in Throwing Bombs, &c.) then $\frac{1}{4}$ of the square Root of the Distance, *in Feet*, will give the Number of *Seconds* taken up in the Flight. The Knowing of which will be of Use in adjusting the Fuse.

II. *Of Projections, when the Object fired at, is above, or below, the Level of the Piece.*

P R O B L E M VII.

The horizontal Distance, and the Angle of Elevation, or Depression of an Object, being given, with the Elevation of the Piece; to find the Impetus, so as to hit the Object.

SOLUTION.

As the Rectangle of the Sine of the Elevation of the Piece above the Object, and the Co-sine of its Elevation

vation above the Horizon, is to the Rectangle under the Radius and the Co-sine of the Object's Elevation, or Depression; so is $\frac{1}{4}$ of the given horizontal Distance of the Object, to the Impetus required (*by Prop. 3. Corol. 2*).

Ex. Let the horizontal Distance of the Object be 5600 Feet, and its Elevation $8^{\circ} : 15'$; and let the Elevation of the Piece be $32^{\circ} : 30'$: Then,

$24^{\circ} : 15'$, Co-ar. of its Sine	—	0. 3865	} to be added
$32^{\circ} : 30'$, Co-ar. of its Co-s.	—	0. 0740	
Radius	—	10. 0000	
$8^{\circ} : 15'$, its Co-sine	—	9. 9955	
1400 F.	—	3. 1461	
4000 F. the Imp. req ^d .	—	2 3. 6021	

P R O B L E M VIII.

The horizontal Distance, and the Angle of Elevation, or Depression of an Object, being given, together with the Impetus; to find the Elevation of the Piece, to hit the Object.

SOLUTION.

As the Radius is to the Tangent of the Object's Elevation, or Depression, so is twice the Impetus to a fourth Number; which add to, or subtract from, the given horizontal Distance, according as the Object is elevated, or depressed: Then say,

As twice the Impetus is to the Sum, or Remainder, so is the Co-sine of the given Elevation, or Depression, to the Co-sine of an Angle; which added to, and subtracted from, the Angle included between the Object and the Zenith (or vertical Point), gives the Double of the Complements of two different Elevations, whereby the Ball may hit the Object (*See Prop. 3*).

Ex. Let the horizontal Distance of the Object be 5600 Feet, and its Elevation $8^{\circ} : 15'$; and let the given Impetus be 4000 F.

Then,

Then, as Radius	—————	10. 0000
is to Tang. $8^{\circ} : 15'$	—————	9. 1613
so is 8000	—————	3. 9031
to — 1160	—————	<u>13. 0644</u>

This added to 5600 gives 6760 :

Therefore, as 8000	—————	3. 9031
is to — 6760	—————	3. 8299
so is Co-f. $8^{\circ} : 15'$	—————	9. 9955
to the Co-f. $33 : 15$	—————	<u>9. 9223</u>

Which, added to, and subtracted from $81^{\circ} : 45'$, gives $115^{\circ} : 00'$, and $48^{\circ} : 30'$, respectively : The Halves of which are $57^{\circ} : 30'$, and $24^{\circ} : 15'$; whose Complements $32^{\circ} : 30'$, and $65^{\circ} : 45'$, are the two Elevations required.

P R O B L E M IX.

The Impetus, and the Angle of Elevation, being given; to find at what Distance the Piece ought to be planted, to hit an Object, whose Distance above, or below, the Level of the Piece, is also given.

SOLUTION.

As the Radius, is to the Sine of twice the given Elevation, so is the Impetus, to half the horizontal Amplitude, at that Elevation (*by Prop. 1*).

And, as the Radius, is to the Co-tangent of the Elevation, so is twice the perpendicular Height, or Depression, of the Object, to a Fourth-Proportional; which take from, or add to, half the horizontal Amplitude, according as the Object is elevated, or depressed; then find a Mean Proportion between the Half-Amplitude, and the Sum, or Remainder; which, added to the said Half-Amplitude, gives the Distance sought (*by Prop. 2*).

Ex. Let the Impetus be 3000 Feet, the Elevation 40° , and the Height of the Object 200 Feet.

Then,

Then, as Radius	—————	10. 0000
to Sine — 80°	—————	9. 9933
so is — 3000	—————	3. 4771
to — 2954	—————	1 3. 4704

And, as Radius	—————	10. 0000
is to Co-t. 40°	—————	10. 0762
so is — 400	—————	2. 6021
to — 477	—————	1 2. 6783

Now, the Difference between the two Numbers above found is 2477 ; whose Log. being added to that of 2954, and the Sum divided by 2, the Quotient will be the Log. of 2705, the required Mean Proportional : Whence the Distance sought comes out 5659 Feet.

III. Of Projections on Planes, inclin'd to that of the Horizon.

P R O B L E M X.

The Inclination of the Plane, and the Elevation and Impetus of the Piece, being known ; to find the Amplitude of the Projection.

SOLUTION.

As the Square of the Co-sine of the Plane's Inclination to the Horizon, is to the Rectangle of the Sine of the Elevation above the Plane and the Co-sine of the Elevation above the Horizon, so is 4 times the Impetus, to the Amplitude of the Projection (*by Prop. 3. Cor. 3*).

Ex. 1. Let the Impetus be 4000 Feet, the Elevation 32° : 30', and the Ascent of the Plane 8° : 15'. Then the Elevation above the Plane will be 24° : 15' ; and the Operation as follows.

8° :

8° : 15', Co-Arith. of its Co-f. —	0. 0045	} to be added
<i>The same repeated</i> ———	0. 0045	
24° : 15' — its Sine ———	9. 6135	
32° : 30' — its Co-f. ———	9. 9260	
16000 F. — its Log. ———	4. 2041	
5658 Feet the Amp. req ^d . —	2 3. 7526	

Ex. 2. Let the Elevation and Impetus be the same as in the last Example; but let the Plane in this Case have a Descent of 8° : 15' (instead of an equal Ascent).

Then the Operation will stand thus,

8° : 25', Co-Ar. of its Co-f. ———	0. 0045	} to be added
<i>The same repeated</i> ———	0. 0045	
40° : 45', its Sine ———	9. 8147	
32° : 20', its Co-sine ———	9. 9260	
16000 F. its Log. ———	4. 2041	
8992 F. Amp. req ^d . ———	2 3. 9538	

P R O B L E M XI.

The Inclination of the Plane, the Elevation of the Piece, and the Amplitude of the Projection, being given; to find the Impetus.

SOLUTION.

As the Rectangle of the Sine of the Elevation above the Plane and the Co-sine of the Elevation above the Horizon, is to the Square of the Co-sine of the Plane's Inclination, so is the given Amplitude, to 4 times the required Impetus (*by Prop. 3. Cor. 3*).

Ex. Suppose the Plane to have an Ascent of 8° : 15', and that the Amplitude thereon, at an Elevation of 32° : 30', is 5658 Feet. Then, the Elevation above the Plane being 24° : 15', we shall have,

24° : 15' Co-Ar. of its Sine ———	0. 3865	} to be added
32° : 30' Co-Ar. of its Co f. ———	0. 0740	
8° : 15' its Co-f. ———	9. 9955	
<i>The same</i> ———	9. 9955	
5658 F. its Log. ———	3. 7526	
16000 F. ———	2 4. 2041	

$\frac{1}{4}$ of which, or 4000 Feet, is the Impetus, req^d.

P R O B L E M XII.

The Inclination of the Plane, the Impetus, and the Amplitude being given, to determine the Elevation.

SOLUTION.

As the Radius is to the Co-sine of the Plane's Inclination, so is the given Distance, on the Plane, to the horizontal Distance corresponding: From which, the Impetus, and the Plane's (or Object's) Elevation or Depression, the Elevation of the Piece may be found, *by Prob. 8.*

Ex. Let the Plane have an Ascent of 8° : 15', and let the given Amplitude thereon be 5658 Feet, supposing the *Impetus* to be 4000 Feet.

Then, as the Radius ———	10. 0000
is to the Co-sine of 8° : 15' —	9. 9955
so is 5658 Feet ———	3. 7526
to — 5600 Feet ———	1 3. 7481

the horizontal Distance of the Place where the Ball impinges: As for the rest of the Operation, it is exactly the same as in the Example to *Prob. 8*; for which Reason it will be needless to repeat it here.

P R O B L E M XIII.

Having the Angle of Elevation, or Depreffion of the Object, together with the Elevation, and Impetus of the Piece; to determine the Time of the Flight.

SOLUTION.

As the Co-fine of the Elevation, or Depreffion of the Object, is to the Sine of the Elevation of the Piece above the Object, fo is Half the fquare Root of the Number of Feet in the *Impetus* given, to the requir'd Number of Seconds in the Flight.

Ex. Suppose the Elevation of the Object to be $8^{\circ} : 30'$; That of the Piece 45° ; and the *Impetus* 3600 Feet.

Then, the fquare Root of 3600 being 60, we have

As the Co-fine $8^{\circ} : 30'$	— Co-Ar.	0. 0048
is to the Sine $36^{\circ} : 30$	—————	9. 7744
fo is 30	—————	1. 4771
to 18,04 Seconds req ^d .	—————	1 1. 2563

IV. *Of the MAXIMA and MINIMA, in the Motion of Projectiles.*

P R O B L E M XIV.

The greatest horizontal Amplitude being given, to find the greatest Amplitude on a Plane whose Inclination to the Horizon is also given.

SOLUTION.

Take half the Angle included between the Plane and the Zenith; the Complement of which is the required Elevation (*by Prop. 3. Cor. 6*). Then fay, as the Tangent of the faid Elevation, is to the Tangent of the Plane's Inclination, fo is the given Amplitude, to the Difference between it and the Amplitude fought (*by Prop. 3. Corol. 7*).

Ex.

Ex. Let the greatest horizontal Range be 8000 Feet, and the Inclination of the Plane $12^{\circ} : 30'$, descending.

Here the Angle included between the Plane and the Zenith, or vertical Point, being $102^{\circ} : 30'$, the Half thereof will be $51^{\circ} : 15'$, and the Elevation $38^{\circ} : 45'$.

Therf. as Tang. $38^{\circ} : 45'$	————	9. 9045
is to Tang. — $12^{\circ} : 30'$	————	9. 3457
so is — — — 8000 F.	————	3. 9031
to ————— 2210 F.	————	3. 3443

Which, added to 8000 (because the Plane descends), gives 10210 Feet, for the true Answer, in this Case.

P R O B L E M XV.

The greatest Amplitude, on an inclin'd Plane, being given, to find the greatest Amplitude, on the Plane of the Horizon.

SOLUTION.

As the Radius, is to the Sine of the Plane's Inclination, so is the given Amplitude, to the Difference between it and the required Amplitude (*by Prop. 3, Corol. 8*).

Ex. Let the Inclination of the Plane be $12^{\circ} : 30'$, descending, and the given Amplitude 10210 Feet.

Then, as Radius	—————	10. 0000
is to S. $12^{\circ} : 30'$	—————	9. 3353
so is — — 10210	—————	4. 0090
to — — — 2210	—————	3. 3443

Which taken from 10210, gives 8000 Feet, for the true Distance sought.

P R O B L E M XVI.

The Impetus, and the perpendicular Height, or Depression, of an Object being given, to find the greatest horizontal Distance at which the Ball can, possibly, hit the Object, and also the Elevation answering thereto.

SOLUTION.

Take the Difference of the two given Quantities, if the Object be elevated; but otherwise, their Sum; then find a Mean Proportional between the Impetus and said Difference, or Sum; the Double of which will be the Distance sought (*by Prop. 3. Cor. 9*).

For the Elevation, it will be, as the Distance thus found is to the Height, or Depression of the Object, so is the Radius to the Tangent of an Angle; which added to, or subtracted from, 90 Degrees, respectively, gives the Double of the required Elevation (*by Prop. 3. Cor. 9*).

Ex. Let the Impetus be 4000 Feet, and the Depression of the Object, below the Level of the Piece, 2210 Feet.

Here we are to take a Mean Proportional between 4000 and 6210; which is $= \sqrt{4000 \times 6210} = 4984$; whose Double 9968 is the required Distance.

Moreover, we have, as 9968	————	3. 9986
is to	———— 2210	3. 3443
so is Radius	————	10. 0000
to Tang.	———— 12° : 30	9. 3457

whence the Elevation appears to be 38° : 45°.

From this Problem, the greatest Amplitude of a Ball, projected from a given Height above the Level of the Horizon, is given.

PROBLEM

P R O B L E M XVII.

Having the horizontal Distance of the Object, together with its perpendicular Height, or Depression, above or below the Level of the Piece; to determine the least Impetus, possible, whereby the Ball may reach the Object; and also the Elevation corresponding.

SOLUTION.

As the horizontal Distance of the Object is to its perpendicular Height or Depression, so is the Radius to the Tangent of an Angle; which, added to, or subtracted from 90 Degrees, gives the Double of the Elevation (*by Prop. 3. Corol. 6*).

And, as the Radius is to the Tangent of the Elevation, so is the given horizontal Distance, to twice the Impetus required (*by Prop. 3. Corol. 11*).

Ex. Let the horizontal Distance be 9968 Feet, and the Distance of the Object below the Level of the Piece 2210 Feet.

Then, as — 9968 —————	3. 9986
is to — 2210 —————	3. 3443
so is Rad. —————	10. 0000
to Tang. $12^{\circ} : 30'$ —————	9. 3457

Therefore the Elevation is $38^{\circ} : 45'$; and it will be

As Radius —————	10. 0000
is to Tang. $38^{\circ} : 45'$ —————	9. 9945
so is — 9968 —————	3. 9986
to — 8000 —————	3. 9031

The Half of which, or 4000 Feet, is the Impetus sought.

P R O B L E M XVIII.

To find the greatest Height a Ball can, possibly, reach in a Perpendicular to the Horizon; the Impetus, and the Distance of the Piece from the said Perpendicular, being given.

SOLUTION.

Find a Third Proportional to the Impetus and half the given Distance, which subtract from the Impetus, and the Remainder will be the Answer (*by Prop. 3. Corol. 10*).

Thus, if the Impetus be 4000 F. and the given Distance 3000 Feet, then it will be, as $4000 : 3000 :: 3000 : 2250$; which taken from 4000, leaves 1750 Feet, for the greatest Height the Ball can possibly reach in the proposed Perpendicular, with that Impetus.



PART IV.

EXHIBITING

A new, and very comprehensive Method for extracting the Roots of Equations in Numbers; by increasing the Dimensions of the unknown Quantity.

IN this Method (as in the common Method by converging Series's) the Value sought is, first of all, to be nearly estimated (either from the Equation itself, or from the Nature of the Problem whence it is derived); and some unknown Quantity (as z) must be assumed to express the Difference between that Value (which we will denote by r) and the true Value required: And then, by substituting $r \pm z$ instead of its Equal, in the given Equation, a new Equation will emerge, affected only with z and known Quantities.

The Equation being thus prepared for a Solution, multiply it into two, or more succeeding Terms of the Series $1 + Az + Bz^2 + Cz^3 + Dz^4 \text{ \&c.}$ (according to the Degree of Exactness Necessary) and let the Coefficients of the homologous Terms of the new Equation hence arising, wherein the Square, and the next, higher Powers of z are concerned, be made equal to Nothing: By which Means the Value of A , \&c. will be determined, and as many Terms of the Equation destroyed, at the same time, as there are Terms in the assumed Multiplier *minus* one. And the Terms involving the superior Powers, which yet remain, being of very small Value, may also be rejected: Whence the Equation will be reduced to a Simple One: From which the Value of z will be found.

Ex. 1. Let the Equation propounded be $x^2 = 50$.

Here, x being something greater than 7, put $7 + z = x$; then the given Equation will become $49 + 14z + z^2 = 50$;

or, $-1 + 14z + z^2 = 0$. This, multiply'd
by $1 + Az + Bz^2$,

$$\text{gives } \left\{ \begin{array}{l} -1 + 14z + z^2 \\ * - Az + 14Az^2 + Az^3 \\ * - Bz^2 + 14Bz^3 + Bz^4 \end{array} \right\} = 0.$$

Whence, by equating the Coefficients of the like Terms, we have $-1 + 14A - B = 0$, and $A + 14B = 0$:

From which Equations A is found $= -\frac{14}{197}$; and B

$$\left(= -\frac{A}{14} \right) = \frac{1}{197}:$$

And, by substituting these Values above,

$$\text{we have } -1 + 14 + \frac{14}{197} \times z + \frac{z^4}{197} = 0.$$

Which, by rejecting the exceeding small Quantity $\frac{z^4}{197}$,

$$\text{becomes } -1 + 14 + \frac{14}{197} \times z = 0.$$

$$\text{Hence } -197 + 2772z = 0;$$

and $z = \frac{197}{2772} = .0710678$, nearly: Which Value is

true to the last Decimal Place: And, if more Terms of the Series $1 + Az + Bz^2 + Cz^3$ &c. had been taken, the Conclusion would have been still exacter in Proportion.

Ex. 2. Suppose the given Equation (when prepared for a Solution) to be $-2 + 5z - z^3 = 0$.

Here, if four Terms of the general Series $1 + Az + Bz^2 + Cz^3$ &c. be taken, and multiply'd by $-2 + 5z - z^3$, our Equation will be chang'd into the following One,

viz.

218 *The* RESOLUTION of EQUATIONS.

Term only, may, by the same Rule, be call'd an Approximation of the First Degree.)

But to obtain an Approximation of the Third Degree, or such an One as shall include three Terms of the original Series, let the given Equation,

$-p + az + bz^2 + cz^3 \&c. = 0$, be now multiply'd by three Terms of the assumed Series $1 + Az + Bz^2 \&c.$: Whence there arises this

$$\text{new Equation} \left\{ \begin{array}{l} -p + az + bz^2 + cz^3 \&c. \\ * -Apz + Aaz^2 + Abz^3 \&c. \\ * * -Bpz^2 + Baz^3 \&c. \end{array} \right\} = 0.$$

Where, by equating the homologous Terms, we have $b + Aa - Bp = 0$, and $c + Ab + Ba = 0$.

Let the former of these Equations be multiply'd by a , and the Latter by p ; and then add the two Products together; so shall

$ab + Aa^2 + pc + Abp = 0$; and consequently $A = -\frac{ab + cp}{aa + bp}$. Whence $z \left(= -\frac{p}{a - pA} \right)$ is likewise given.

To have an Approximation of the Fourth Degree, four Terms of the Series $1 + Az + Bz^2 \&c.$ must be taken, for a Multiplier: By which means the Equation given will be

$$\text{trans-} \left\{ \begin{array}{l} -p + az + bz^2 + cz^3 + dz^4 \&c. \\ \text{form'd} * -Apz + Aaz^2 + Abz^3 + Acz^4 \&c. \\ \text{to} * * -Bpz^2 + Baz^3 + Bbz^4 \&c. \\ * * * -Cpz^3 + Caz^4 \&c. \end{array} \right\} = 0.$$

Here we have

$$B = \frac{Aa}{p} + \frac{b}{p},$$

$$C = \frac{Ba}{p} + \frac{Ab}{p} + \frac{c}{p},$$

$$0 = Ca + Bb + Ac + d,$$

From whence, exterminating C , there arises

$$\frac{Ba}{p} + \frac{Ab}{p} + \frac{c}{p} + \frac{Bb}{a} + \frac{Ac}{a} + \frac{d}{a} = 0,$$

$$\text{or } \frac{a}{p} + \frac{b}{a} \times B + \frac{b}{p} + \frac{c}{a} \times A + \frac{c}{p} + \frac{d}{a} = 0.$$

Which, by substituting the Value of B,

$$\text{becomes } \frac{a}{p} + \frac{b}{a} \times \frac{Aa}{p} + \frac{b}{p} + \frac{b}{p} + \frac{c}{a} \times A + \frac{c}{p} + \frac{d}{a} = 0;$$

$$\text{that is, } \frac{aa}{pp} + \frac{2b}{p} + \frac{c}{a} \times A + \frac{ab}{pp} + \frac{bb}{ap} + \frac{c}{p} + \frac{d}{a} = 0.$$

Multiply the Whole by ap^2 ; then will $a^3 + 2abp + cpp$
 $\times A + a^2b + b^2p + acp + dp^2 = 0$:

$$\text{Whence } A \text{ is found} = -\frac{aab + ac + bb \times p + dpp}{a^3 + 2abp + cpp};$$

$$\text{And } z = \frac{p}{a - pA}, \text{ as in the preceding Case.}$$

By the same Method, an Approximation of any higher Degree, to take in as many Terms of the proposed Series as you please, may be derived.

For, it is evident, from above, that in all Cases whatever,

$$B \text{ will be } = \frac{Aa}{p} + \frac{b}{p},$$

$$C = \frac{Ba}{p} + \frac{Ab}{p} + \frac{c}{p},$$

$$D = \frac{Ca}{p} + \frac{Bb}{p} + \frac{Ac}{p} + \frac{d}{p},$$

$$E = \frac{Da}{p} + \frac{Cb}{p} + \frac{Bc}{p} + \frac{Ad}{p} + \frac{e}{p},$$

$\mathcal{E}c. \quad \mathcal{E}c.$

where the last Value is, always, equal to Nothing.

Therefore

$$\text{Therefore, by making } \left\{ \begin{array}{l} Q = \frac{a}{p} \\ R = \frac{aQ + b}{p} \\ S = \frac{aR + bQ + c}{p} \\ T = \frac{aS + bR + cQ + d}{p} \\ V = \frac{aT + bS + cR + dQ + e}{p} \\ \text{\&c.} \end{array} \right.$$

$$\text{and } \left\{ \begin{array}{l} q = \frac{b}{p} \\ r = \frac{aq + c}{p} \\ s = \frac{ar + bq + d}{p} \\ t = \frac{as + br + cq + e}{p} \\ v = \frac{at + bs + cr + dq + f}{p} \\ \text{\&c.} \end{array} \right.$$

the Value of A will be determined, by dividing the last of these Quantities $q, r, s, t, \text{\&c.}$ by the corresponding Quantity of the upper Series $Q, R, S, \text{\&c.}$ and changing the Sign of the Quotient.

$$\text{Whence we get } \frac{p}{a + p \times \frac{q}{Q}}, \frac{p}{a + p \times \frac{r}{R}}, \frac{p}{a + p \times \frac{s}{S}}, \text{\&c.}$$

$$\text{or, } \frac{1}{\frac{a}{p} + \frac{q}{Q}}, \frac{1}{\frac{a}{p} + \frac{r}{R}}, \frac{1}{\frac{a}{p} + \frac{s}{S}}, \text{\&c. for so many}$$

different Values of z ; whereof each is more exact than the preceding One.

These

These Equations are easily derived from Those above, expressing the Relation of the Quantities A, B, C, D, &c.

For, by writing Q and q, instead of their Equals, in the first of those Equations $(B = \frac{Aa}{p} + \frac{b}{p})$ it becomes $B = QA + q$.

Which Value of B, wrote in the 2^d Equation,

$$C = \frac{Ba}{p} + \frac{Ab}{p} + \frac{c}{p}, \text{ gives}$$

$$C = \frac{aQA}{p} + \frac{aq}{p} + \frac{Ab}{p} + \frac{c}{p} = RA + r; \text{ by substituting R and r in the Room of their Equals.}$$

Moreover the third Equation, by substituting for C and B, becomes $D = \frac{aRA}{p} + \frac{ra}{p} + \frac{bQA}{p} + \frac{bq}{p} + \frac{Ac}{p} + \frac{d}{p} = SA + s$.

After the very same manner $E = TA + t$, $F = VA + v$, &c. And, by putting all these several Values, successively, equal to Nothing, we have $-\frac{q}{Q}$, $-\frac{r}{R}$, $-\frac{s}{S}$, &c. for so many different Approximations of the Value A. Which being substituted in the general Equation $z = \frac{p}{a - pA}$, the very Expressions before given, for the Value of z, are obtained.

It will be proper, now, to shew the Use of the several Approximations, or Theorems, derived above, by a few Examples.

In the first Place, then, let the Equation given be
 $x^3 = 10.$

This Equation, by making $2 + z = x$, will be transformed to $-2 + 12z + 6z^2 + z^3 = 0$: Which being compared with the general Equation

$-p + az + bz^2 + cz^3 + dz^4 \&c. = 0$,
 we have $p = 2$, $a = 12$, $b = 6$, $c = 1$, $d = 0$, $\&c.$

Therefore (by the first of the three Approximations, at p. 217) $-A\left(=\frac{b}{a}\right)$ is found $= \frac{1}{2}$.

Whence $z\left(=\frac{p}{a-Ap}\right)$ comes out $= \frac{1}{13} = 0.1538$, nearly.

But, according to the second Approximation, or Theorem, the Value of $-A\left(=\frac{ab + cp}{aa + bp}\right)$ will be $= \frac{72 + 2}{144 + 12} = \frac{37}{78}$: And consequently that of $z\left(=\frac{p}{a-Ap}\right) = \frac{78}{505} = 0.15445$.

Lastly, the Value of $-A$, according to the third Approximation being $= \frac{a^2b + ac + bb \times p + dp^2}{a^3 + 2abp + cpp} = \frac{144 \times 6 + 96}{144 \times 12 + 144 \times 2 + 4} = \frac{36 \times 6 + 24}{36 \times 14 + 1} = \frac{48}{101}$, the corresponding Value of z will therefore come out $= \frac{101}{654} = 0.154434$: Which Number is true in all its Places.

The very same Conclusions will likewise be brought out, from the general Solution.

For,

For, p being here $= 2$, $a = 12$, $b = 6$, $c = 1$,
 $d = 0$. &c. (as before).

we have $Q\left(\frac{a}{p}\right) = 6$,

$$R\left(\frac{aQ + b}{p}\right) = \frac{12 \times 6 + 6}{2} = 39,$$

$$S\left(\frac{aR + bQ + c}{p}\right) = \frac{505}{2},$$

&c.

Also, $q\left(\frac{b}{p}\right) = 3$,

$$r\left(\frac{aq + c}{p}\right) = \frac{37}{2},$$

$$s\left(\frac{ar + bq + d}{p}\right) = 120,$$

&c.

Therefore $\frac{q}{Q} = \frac{1}{2}$, $\frac{r}{R} = \frac{37}{78}$, $\frac{s}{S} = \frac{240}{505} = \frac{48}{101}$; and

consequently $\frac{\frac{1}{a} + \frac{q}{p}}{\frac{1}{Q} + \frac{q}{p}} = \frac{1}{6 + \frac{1}{2}} = \frac{2}{13}$, &c. as before.

For a second Example, suppose $10z - z^3 = 2$, or
 $-2 + 10z - z^3 = 0$.

In this Case $p = 2$, $a = 10$, $b = 0$, $c = -1$, $d = 0$, &c.
 And therefore, for an Approximation of the fourth

Degree, we have $-A\left(\frac{aab + ac + bb \times p + dpp}{a^3 + 2abp + cpp}\right) =$

$$\frac{-20}{1000 - 4} = \frac{-5}{249}; \text{ and consequently } z\left(\frac{p}{a - pA}\right)$$

$$= \frac{249}{1240} = 0.2008045; \text{ which Value is true to the}$$

last Figure.

For

224 *The* RESOLUTION of EQUATIONS.

For a third Example, let there be given the Equation

$$x^3 + 36x^2 + 432x = 2272.$$

Here the Value of x appears to be about 4; let therefore $4 + z$ be wrote for x ; whence our Equation is reduced to

$$96 + 768z + 48z^2 + z^3 = 0.$$

Which being compared with

$$-p + az + bz^2 + cz^3 + dz^4 \&c. = 0,$$

we have, in this Case,

$$p = -96, a = 768, b = 48, c = 1, d = 0, \&c.$$

$$\text{Or, } \frac{a}{p} = -8, \frac{b}{p} = -\frac{1}{2}, \frac{c}{p} = -\frac{1}{96}, \&c.$$

$$\text{Hence } Q\left(\frac{a}{p}\right) = -8,$$

$$R\left(\frac{aQ}{p} + \frac{b}{p}\right) = 64 - \frac{1}{2} = \frac{127}{2},$$

$$S\left(\frac{aR}{p} + \frac{bQ}{p} + \frac{c}{p}\right) = -\frac{48385}{96};$$

$\&c.$

$$\text{Also } q\left(\frac{b}{p}\right) = -\frac{1}{2},$$

$$r\left(\frac{aq}{p} + \frac{c}{p}\right) = 4 - \frac{1}{96} = \frac{383}{96},$$

$$s\left(\frac{ar}{p} + \frac{bq}{p} + \frac{d}{p}\right) = -\frac{380}{12},$$

$\&c.$

$$\text{Therefore } \frac{q}{Q} = \frac{1}{16}, \frac{r}{R} = \frac{383}{127 \times 48} = \frac{383}{6096}, \frac{s}{S} = \frac{380 \times 8}{48385} = \frac{3040}{48385};$$

$$\text{From whence } z = -\frac{16}{127} = -0.12598, \text{ nearly;}$$

$$\text{or, } z = -\frac{6096}{48385} = -0.1259894, \text{ more nearly,}$$

$$\text{or, } z = -\frac{9677}{76808} = -0.1259894802, \text{ still nearer.}$$

Lastly,

Lastly, let there be given the Equation

$$\frac{x^2}{2} - \frac{x^4}{2 \cdot 3 \cdot 4} + \frac{x^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \frac{x^8}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \mathfrak{E}^c = \frac{1}{2}.$$

Then, making $z = x^2$, \mathfrak{E}^c . we have

$$-1 + z - \frac{z^2}{12} + \frac{z^3}{12 \times 30} - \frac{z^4}{12 \times 30 \times 56} \mathfrak{E}^c = 0.$$

Here $p = 1$, $a = 1$, $b = -\frac{1}{12}$, $c = \frac{1}{12 \times 30}$, $d =$

$$-\frac{1}{12 \times 30 \times 56} : \mathfrak{E}^c.$$

And, by substituting these Values in the third General Approximation (*vid. p. 219*), we have — A

$$\left(= \frac{aab + ac + bb \times p + dpp}{aaa + 2abp + cpp} \right)$$

$$= \frac{-\frac{1}{12} + \frac{1}{12 \times 30} + \frac{1}{12 \times 12} - \frac{1}{12 \times 30 \times 56}}{1 - \frac{2}{12} + \frac{1}{12 \times 30}} = -$$

$$\frac{30 \times 56 - 56 - 140 + 1}{10 \times 30 \times 56 + 56} = -\frac{1485}{16856}.$$

$$\text{Therefore } z \left(= \frac{p}{a - pA} \right) = \frac{16856}{15371} = 1.09661;$$

$$\text{and } x (= \sqrt{z}) = 1.04719.$$

After the same manner the Roots of other Equations may be approximated: But I shall here shew, how the general Theorems themselves may be rendered more commodious, for certain particular Cases, by Means of a proper Transformation.

It is known, if two Quantities be, respectively, increased, or decreased by two other, small, Quantities, nearly in the same Ratio with the two First, that the Sums, or Differences will still be in the same Ratio with the two first Quantities, very near.

Q

Wherefore

Wherefore, seeing the Numerator of the Fraction $\frac{ab + cp}{aa + bp}$, expressing the 2^d. General Value of $-A$. (*vid.* p. 218) is in proportion to the Denominator, nearly, as b to a , or cp to $\frac{acp}{b}$; let cp be therefore taken from the Numerator, and $\frac{acp}{b}$ from the Denominator; agreeable to the above Observation: By which Means the Fraction itself will be transformed to $\frac{ab}{aa + bp - \frac{acp}{b}} =$

$$\frac{b}{a + \frac{b}{a} - \frac{c}{b} \times p} = \frac{b}{a + rp}; \text{ supposing } r = \frac{b}{a} - \frac{c}{b}.$$

And, in the very same Manner the Fraction $\frac{aab + ac + bb \times p + dpp}{aaa + 2abp + cpp}$, expressing the third Value of $-A$, is changed to $\frac{aab + ac + bb \times p}{aaa + 2abp + cpp - \frac{adpp}{b}} =$

$$\frac{aab + ac + bb \times p}{a^3 + 2abp + sp^2} \left(\text{by putting } s = c - \frac{ad}{b} \right).$$

But this last Value is still capable of a further Reduction: For, the Ratio of the Numerator to the Denominator being That of $\frac{bsp}{r}$ to $\frac{asp}{r} + sp^2$, nearly (as appears from the preceding Case) let, therefore, the former of these Quantities be subtracted from the Numerator, and the latter from the Denominator: Whence the Fraction itself becomes

$$\frac{a^2b + ac + bb \times p - \frac{bsp}{r}}{a^3 + 2abp - \frac{asp}{r}} = \frac{ab + c + \frac{bb}{a} - \frac{bs}{ra} \times p}{a^2 + 2b - \frac{s}{r} \times p} =$$

$$= \frac{ab + c + \overline{bw} \times p}{aa + b + \overline{aw} \times p}; \text{ by putting } w = \frac{b}{a} - \frac{s}{ar}$$

$$\left(= \frac{b}{a} - \frac{cb - ad}{bb - ac} \right)$$

Now, to exemplify the Use of the Theorems thus transformed, let the Equation $96 + 768z + 48z^2 + z^3 = 0$ (given at p. 224) be here resumed :

Then, in this Case, p being $= -96$, $a = 768$, $b = 48$, $c = 1$, $d = 0$, &c. we have $r \left(= \frac{b}{a} - \frac{c}{b} \right) = \frac{1}{24}$, $s \left(= c - \frac{ad}{b} \right) = 1$, and $w \left(= \frac{b}{a} - \frac{s}{ar} \right) = \frac{1}{32}$.

Whence, according to the first Approximation,

$$-A \left(= \frac{b}{a + rp} \right) = \frac{48}{768 - 4} = \frac{12}{191}.$$

And, according to the Second,

$$-A \left(= \frac{ab + c + \overline{bw} \times p}{aa + b + \overline{aw} \times p} \right) = \frac{768 \times 48 + 1 + \frac{1}{2} \times -96}{768 \times 768 + 48 + 24 \times -96}$$

$$= \frac{768 - 5}{768 \times 16 - 72 \times 2} \text{ (by dividing every Term by 48)}$$

$$= \frac{763}{12144}.$$

Hence the Value of $z \left(= \frac{1}{\frac{a}{p} - A} \right)$ comes out $= -$

$$\frac{191}{1516} = -0, 1259894, \text{ nearly; or equal to } -\frac{12144}{96389}$$

$$= -0. 1259894802, \text{ more nearly.}$$

These Conclusions agree with Those before given, by the former Method. But the last general Approximations, containing the fewest Dimensions of the Quantity p , will commonly be found to have the Advantage, in point of Expedition, when the Value of that Quantity consists of several Decimal

Q 2

Places,

228 *The RESOLUTION of EQUATIONS.*

Places, and also when the Coefficients a, b, c, d , of the Powers of the unknown Quantity z , are related to each other according to some known Law.

Of this Kind are the Coefficients of such Series's as arise in extracting the Roots of pure Powers; and in these Cases the general Theorems, or Equation themselves, are capable of being rendered still more commodious.

Let there be assumed the Equation $x^n = k$ (which includes all the Cases of pure, or simple Powers, according to the Value of the Exponent n).

Then, by assuming r nearly equal to x , and making $r \times \overline{1+z} = x$, we shall have $r^n \times \overline{1+z}^n = k$;

and therefore $-\frac{k}{r^n} + \overline{1+z}^n = 0$; that is

$$-\frac{k}{r^n} + 1 + nz + \frac{n}{1} \times \frac{n-1}{2} z^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} z^3$$

$$\&c. = 0; \text{ or lastly } -p + z + \frac{n-1}{2} z^2 + \frac{n-1}{2} \times$$

$$\frac{n-2}{3} z^3 + \&c. = 0; \text{ by dividing the Whole by } n, \text{ and}$$

$$\text{putting } p = \frac{k - r^n}{nr^n}.$$

Here (by a Comparifon with the general Equation

$$-p + az + bz^2 \&c. = 0) \text{ we have } a=1, b=\frac{n-1}{1}, c=$$

$$\frac{n-1}{2} \times \frac{n-2}{3}, d = \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}, \&c.$$

$$\text{Theref. } r \left(\frac{b}{a} - \frac{c}{b} \right) = \frac{n-1}{2} - \frac{n-2}{3} = \frac{n+1}{6},$$

$$s \left(c - \frac{ad}{b} \right) = \frac{n-2}{3} \times \frac{n-1}{2} - \frac{n-3}{4} = \frac{n-2}{3} \times \frac{n+1}{4},$$

$$w \left(\frac{b}{a} - \frac{s}{ar} \right) = \frac{n-1}{2} - \frac{n-2}{2} = \frac{1}{2},$$

Whence,

Whence, for an Approximation of the third Degree,

$$-A \left(= \frac{b}{a+rp} \right) = \frac{\frac{1}{2} \times n-1}{1+n+1 \times \frac{1}{6} p};$$

$$\text{and } z \left(= \frac{p}{a-Ap} \right) = \frac{p+n+1 \times \frac{1}{6} p^2}{1+2n-1 \times \frac{1}{3} p}.$$

But, for an Approximation of the fourth Degree,

$$-A \left(= \frac{ab+c+bw \times p}{aa+b+aw \times p} \right) = \frac{\frac{1}{2} \times n-1 + n-1 \times 2n-1 \times \frac{1}{12} p}{1+\frac{1}{2} np};$$

$$\text{and } z = \frac{p + \frac{1}{2} np}{1 + \frac{2n-1}{2} \times p + \frac{2n-1 \times n-1}{12} \times pp}.$$

Hence it is manifest that the Root x , of the proposed Equation $x^n = k$, is equal to

$$r + \frac{rp \times 1 + n-1 \times \frac{1}{6} p}{1 + 2n-1 \times \frac{1}{3} p}, \text{ nearly};$$

$$\text{Or, equal to, } r + \frac{rp \times 1 + \frac{1}{2} np}{1 + \frac{2n-1}{2} \times p + \frac{2n-1 \times n-1}{12} \times p^2},$$

more nearly.

But both these Expressions, in Cases where p is a proper Fraction, will be better adapted to practice by making $\frac{nr^n}{k-r^n} = v \left(= \frac{1}{p} \right)$, and substituting $\frac{1}{v}$ for p , its Equal: Whence (after proper Reduction)

$$x = r + \frac{r}{v} \times \frac{6v+n+1}{6v+4n-2}, \text{ nearly};$$

$$\text{Or, } x = r + \frac{r \times 2v+n}{v \times 2v+2n-1 + \frac{1}{6} \times 2n-1 \times n-1},$$

more nearly.

230 *The* RESOLUTION of EQUATIONS.

To shew now the Use and great Exactness of these last Approximations, by an Example, let it be proposed to extract the square Root of 441. Here the general Equation $x^n = k$, becoming $x^2 = 441$, we have $n = 2$, $k = 441$, and $v \left(= \frac{nr^n}{k-r^n} \right) = \frac{800}{41}$; supposing r to be assumed $= 20$.

Therefore, by the first Approximation,

$$x = 20 + \frac{41}{40} \times \frac{1641}{1682} = 20 + \frac{67281}{67280} = 21 + \frac{1}{67280}, \text{ nearly.}$$

$$\text{And, by the Second, } x = 20 + \frac{2758480}{2758481} = 21 - \frac{1}{2758481}, \text{ more nearly.}$$

Again, let there be given the Equation $x^3 = 500$: Then, assuming $r = 8$, we have $v \left(\frac{nr^n}{k-r^n} \right) = \frac{3 \times 512}{-12} = -128$.

Hence, by the first Approximation,

$$x = 8 - \frac{1}{16} \times \frac{-768 + 4}{-768 + 10} = 8 - \frac{191}{3032} = 7,93700527, \text{ nearly:}$$

$$\text{And, by the Second, } x = 8 - \frac{8 \times 253}{128 \times 251 + \frac{1}{3}} = 8 - \frac{6072}{96389} = 7.9370052599, \text{ more nearly.}$$

All the different Approximations hitherto delivered were, originally, derived by multiplying the given Equation into a certain Number of Terms of the assumed Series $1 + Az + Bz^2 + Cz^3 \&c$. But there are other Methods (though, perhaps, none so general) by which the second, third, &c. Dimensions of the unknown Quantity may, in like sort, be destroyed (without assum-

ing

ing any Series) and from thence Value of that Quantity approximated, to what Degree of Exactness you please.

Let there, for Instance, be given the Equation,
 $2z + z^2 = 1$, or $z^2 = 1 - 2z$:

Then, by squaring both Sides thereof, there arises
 $z^4 = 1 - 4z + 4z^2$.

And if, instead of the last Term $4z^2$, its Equal, $1 - 2z$, be substituted, you will have $z^4 = 5 - 12z$:
 This, squared, gives $z^8 = 25 - 120z + 144z^2 = 169 - 408z$, by substituting for z^2 , as before.

Here, rejecting z^8 (on Account of its Smallness in Comparison of the other Terms) we have $408z = 169$, and therefore $z = \frac{169}{408} = 0,414213$, nearly; which is true to the last Figure, inclusive. But, if you would have the Answer still nearer the Truth, let the above Equation $z^2 = 1 - 2z$ be, either; multiply'd, again, by itself, or into some one of the preceding Ones. Thus, if it be multiplied into $z^4 = 5 - 12z$, you will have

$$z^{12} = 845 - 4068z + 4896z^2 = 5741 - 13860z:$$

Where, z^{12} being rejected, z is found $= \frac{5741}{13860}$
 $= 0.41421356$, &c.

Again, if there be given the Equation $z^3 = 3z - p$; then, by squaring both Sides thereof, we have

$$z^6 = 9z^2 - 6pz + p^2:$$

And therefore $z^7 = 9z^3 - 6pz^2 + p^2z = -6pz^2 + pp + 27xz - 9p$; by writing $27z - 9p$ instead of its Equal $9z^3$.

Now, to the triple of this last Equation, let the 2^d. Equation, multiply'd by $2p$, be added:

232 *The* RESOLUTION of EQUATIONS.

By which Means z^2 will be exterminated, and you then will have $3z^7 + 2pz^6 = 81 - 9pp \times z - 27p + 2p^3$.

Whence (rejecting $3z^7 + 2pz^6$) the Value of z is found

$$\frac{27 - 2pp \times p}{9 - pp \times 9}, \text{ nearly.}$$

Various other Expedients might be used, to exterminate the 2^d , 3^d , &c. Powers of the unknown Quantity; but what is already delivered may suffice.



PART

PART V.

GIVING

*Some Account of the Nature of FLUXIONS;
together with the Investigation of the fun-
damental Rules.*

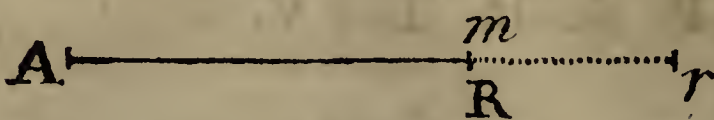
I. **I**N the Doctrine of Fluxions all Kinds of Magnitudes are considered as generated by the continual Motion of some of their Bounds or Extremes; as a Line by the Motion of a Point; a Surface by the Motion of a Line; and a Solid by the Motion of a Surface. So likewise Time may be consider'd as represented by a Line, increasing uniformly by the Motion of a Point: And, as Quantities of all Kinds whatever are capable of Increase and Decrease, They may be represented, in like manner, by Lines, Surfaces, or Solids, conceived to be generated by Motion.

2. Every Quantity thus generated is call'd a *Fluent*, or *Flowing Quantity*: And the Magnitude by which any Flowing Quantity would be uniformly increased, in a given Time, with the generating Celerity at any proposed Position, or Instant * (was it from thence to continue invariable) is the *Fluxion* of the said Quantity at that Position, or Instant.

3. Thus,

* Some Authors define the generating Celerity itself (and not the Magnitude it would produce) to be the Fluxion; but use that Magnitude as the Measure of the said Celerity or Fluxion: Which is, in effect, coming to the same Thing.

3. Thus, let a Point m be conceived to move from A , and thereby generate a Right-line Am , with a Motion any how regulated ; and suppose the Celerity thereof, at any proposed Position R , to be



such, as would, (was it to continue invariable from that Position) be sufficient to describe, or pass uniformly over the Distance Rr , in the given Time allowed for the Fluxion : Then will the said Distance Rr truly express the required Fluxion of the Flowing Line Am , in that Position.

4. It appears from hence, that, when the generating Motion is uniform, the Fluxion, and the Increment *actually* described in the given Time, are one and the same Thing : But, if the Velocity continually increases, or decreases, the Fluxion must then be either less, or greater than the said Increment, or the Space *actually* described : Since an Increase of the Velocity must necessarily cause an Increase in the Distance gone over, and *vice versa*.

Although, in Forming a just and distinct Conception of the Nature, and Quantity of a Fluxion, the Consideration of Time is, absolutely, necessary (on which, even, our Ideas of Velocity depend), yet in the Business and Application of Fluxions, it is not always requisite, that some, vulgar (or common) Measure of Time (as a Second, Minute, Hour, &c.) should be propounded for, the Production of Fluxions of the Quantities under Consideration. A Line generated by the uniform Motion of a Point, it is observed above, may be taken as a proper Representative, or Measure of Time : And that Interval of Time (be it what it will) wherein the Line, so generated, is augmented by any Length, or Fluxion, assigned, may be taken as the Time understood in the Definition, allowed for the Production of the Fluxions of all other Quantities that have any Relation to, or Dependence upon, the said uniformly-generated Line. And these Fluxions themselves, by means of the said Relation and the given Length, or Fluxion, may be also truly exhibited, independent of any particular, known Measure of Time ; as will, hereafter, be fully made to appear.

5. It appears moreover, from the above Definition, that Quantities, which flow, or increase together, so as to continue, *always*, in a constant Ratio, have their Fluxions, likewise, in *the same* constant Ratio.

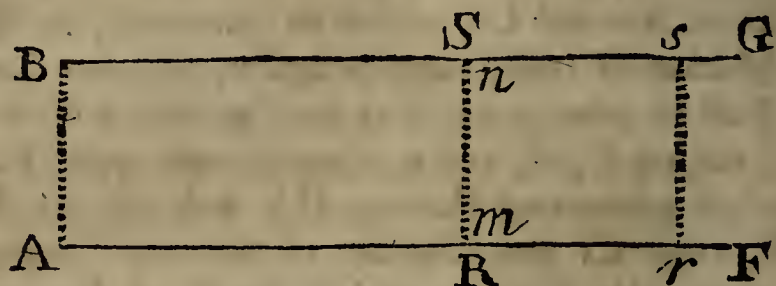
To illustrate This by a particular Example, suppose two Lines, Am and Cn , to be so generated, by the uniform Motion of two Points m and n , that the Latter of them shall be *always* equal to the Double of the former; Then, taking R , S , and r , s as cotemporary Positions of the said generating Points, CS will be the Double of AR , and Cs the Double of Ar , by supposition; whence Ss , the Fluxion of CS , must of Consequence be the Double of Rr , the Fluxion of AR .



It is equally plain, on the other hand, that if the Ratio of the Fluents Am , Cn is variable, That of the Fluxions must also vary.—Thus, if, while the Point m continues to move uniformly on, at the Rate of one Inch (Foot, Yard, &c.) in a Second of Time, the Motion of the other Point n be so regulated that the Number of Inches (Feet, Yards, &c.) in the flowing Line Cn generated thereby, may be *always* equal to the Square of the Number of Those in Am described by the former Point m ; then, in this Case, it is manifest, that the Ratio of the Fluents Am , Cn is a variable One; and that the Ratio of the Fluxions varies also; seeing the Distances 1, 4, 9, 16, 25, &c. described in 1, 2, 3, 4, 5, &c. Seconds of Time, by the Point n , increase much faster in Proportion than 1, 2, 3, 4, 5, &c. the corresponding Distances gone over by the other Point m , moving uniformly.

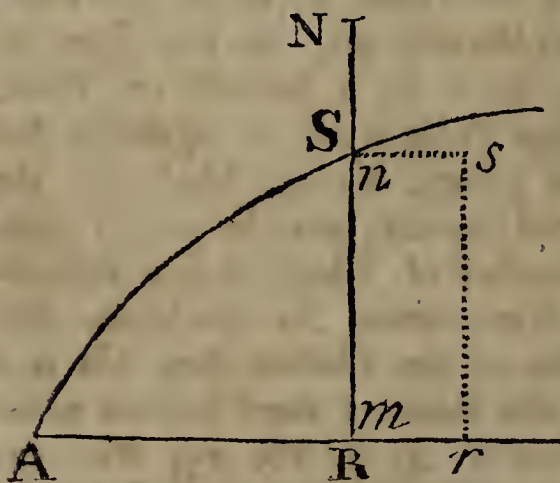
Hitherto

6. Hitherto Regard has been had to the Fluxions of Lines: But the Fluxions of Superficies and Solids are considered in the same manner, and are comprehended with equal Facility. Let a given Right-line mn



be conceived to move parallel to itself, with an equable Motion, from the Position AB , and thereby generate the flowing Rectangle $AmnB$; let also the Distance Rr be taken (*as above*) to express the Fluxion of the Base Am , and let the Rectangle $RrsS$ be completed: Then, this Rectangle being the Space that is uniformly described by the generating Line mn , in the Time that Am is in like manner increased by Rr , it will therefore be the true Fluxion of the flowing Rectangle An , by the Definition. *Art. 2.*

7. The Generation, and the Fluxion of any triangular, or curvilinear, Space ASR , are conceived in



like manner; by supposing a Right-line mN to be carry'd along, continually parallel to itself, so that the intercepted Part Thereof mn (which is itself a variable Quantity) may pass over, and thereby generate, the Space ASR propounded. And the Fluxion

of the Space thus generated, if Rr be taken as the Fluxion of the Base (or Abscissa) Am , will be truly expressed by the Rectangle (Rs) under Rr and RS ; as we shall have Occasion to shew more at large hereafter.

8. From

8. From what has been thus far delivered, it will not be difficult to form a just Idea of the Fluxion of a Solid: But it is time we now come to shew the manner of determining the Fluxions of Algebraic Quantities; by means whereof all Others, of what Kind soever, are explicable: In order to which it will be requisite, first of all, to premise the following Observations.

1. *That, the final Letters u, w, x, y, z , of the Alphabet are usually put for variable Quantities; and the initial Letters a, b, c, d , &c. for invariable Ones: Thus, the variable Base Am of the flowing Rectangle $AmnB$ (in Art. 6.) may be represented by x , and the invariable Altitude mn , by a .*

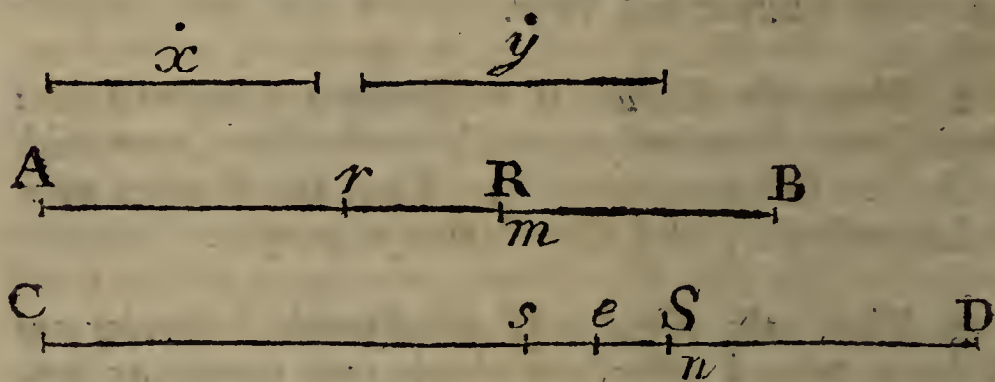
2. *That, the Fluxion of a Quantity represented by a single Letter is commonly expressed by the same Letter with a Dot, or Full-point, over it: Thus the Fluxion of x is denoted by \dot{x} ; and the Fluxion of y by \dot{y} .*

3. *That, the Fluxions of all Quantities (having any Relation to each other) are always to be taken, as contemporaneous, or such as may be generated together, with their respective Celerities, in one and the same Time.*

PROPOSITION I.

9. *The Fluxion of any variable Algebraic Quantity being given, 'tis proposed to find the Fluxion of the Square of that Quantity.*

Let two Points m and n be supposed to move, at the same Time, from two other, fix'd, Points A and C , along the Right-lines AB and CD , in such a manner, that the Measure of the Distance Cn , described by the Latter, may be, *always*, equal to the Square of the contemporary Distance Am described by the former Point m , moving with an equable Celerity.



Moreover, let R and S be any two contemporary Positions of the said Points; and, supposing the described Distances AR and CS to be denoted by x and y , let the Lines \dot{x} and \dot{y} be taken to represent the Spaces that *would be* uniformly passed over, in the same given Time, with the Celerities of the said Points at R and S : So shall those Lines express the Fluxions of the variable Quantities Am and Cn , when the generating Points m and n arrive at the foresaid contemporary Positions R and S (*by the Definition, Art. 2*).

Furthermore, if r and s be considered as any other corresponding Places of the proposed Points, and the Interval rR be denoted by v ; then, AR being $= x$, and $Ar = x - v$, we shall have $CS (= y) = x^2$, and $Cs = \overline{x - v}^2$, *by Hypothesis*; and consequently $Ss (= CS - Cs) = 2xv - vv$.

From

From which it appears, that, while the former Point m moves, uniformly, over the Distance v , the other Point n passeth over a Space expressed by $2xv - vv$.

But this last Distance, since the Velocity of the generating Point n increases continually (*see Art. 5.*) is less than the Space that would be uniformly described, in the same Time, with the Velocity at S ; and greater than That which would be described with the Velocity at s ; and, therefore, is equal to, and may be taken to express, the Space which *might be* uniformly gone over by the Celerity at some intermediate Point e , between s and S , in the same Time.

Therefore, seeing the Distance $(2xv - vv)$ that might be described with the Celerity at the said intermediate Point e , is to the Distance (v) described by m , in the same Time, as $2x - v$ to Unity, it is evident that the said (mean) Celerity at e , must be to the Celerity of m , in the same Ratio; and consequently, that, in the Time the Point m would move over the given Distance x , the other Point n , with its Velocity at e , would describe the Distance $2xx - vx$: Since the Spaces described in equal Times, by uniform Motions, are known to be as the Velocities of the said Motions.

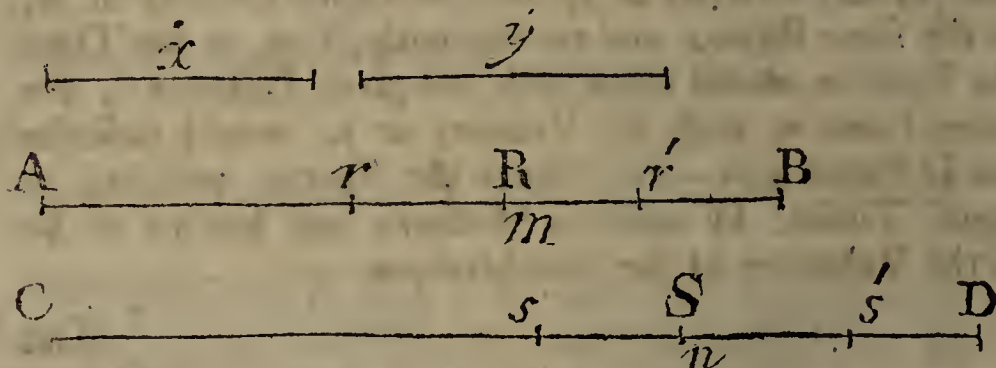
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The above Method of Investigation hath been represented, as bearing a near Affinity to the Method of Prime and Ultimate Ratios: Against which so many Objections have been started, by the celebrated Author of the Analyst, as to embarrass and stagger a great Number of Persons; who are not well apprized how far the said Objections are justifiable, nor wherein their main Force consists. It is not my Design to take Part in a Dispute, on which enough hath been already said by Others: Though, that the Method itself is perfectly Scientific, I believe no One, that understands it, will deny; but whether the great Inventor has therein expressed himself with all the Caution and Accuracy he was capable of, is another Question. Had he call'd That a LIMITING-RATIO which he names an Ultimate-One, the ingenious Author above mention'd might not, perhaps, have found Room for his Ghosts of departed Quantities. However, be this as it will,
those

This being determined, let r be now supposed to coincide with R , and s with S , by means of the Arrival of the generating Points at R and S ; then e , being always between s and S , will likewise coincide with S ; and the foresaid Distance $2xx - vx$, that might be uniformly described with the Velocity at e (now at S) will become, barely, $2xx$; which (*by Art. 2*) is equal to (y) the Fluxion of Cn or x^2 . From whence it appears, that the Fluxion of the Square of any variable, or flowing Quantity is found by multiplying twice the Root, or Quantity itself, into its Fluxion.

The same otherwise.

10. Let the Distance Ar be denoted by u , and let other Things remain as before: Then, CS being =



xx , and $Cs = uu$, by *Supposition*, the Distance sS , described in the same Time with rR ($= x - u$) will therefore be truly expressed by $xx - uu$, or its Equal $\overline{x + u} \times \overline{x - u}$. Which Distance, as the Velocity of n continually increases, must be greater than That which would be uniformly described, with the Celerity at s , in the same Time. Whence it is evident, that the Velocity

those Objections have nothing at all to do with our Method of Investigation given above. The pressing (and only) Difficulty, in the Business of ultimate Ratios, consists, it is known, in considering the Values of the Quantities compared, in their ultimate State, wherein their Ratio is supposed to be taken:

For

Velocity at s is to the, uniform, Velocity of m , in a less Ratio than That of $\overline{x+u} \times \overline{x-u}$ to $x-u$, or of $\overline{x+u}$ to Unity, or lastly, of $\overline{x+u} \times \overline{x}$ to \overline{x} (because the Velocities of uniform Motions are as the Spaces described in equal Times). Therefore, since the uniform Velocity of m (measured by the Distance moved over in a given Time) is defined by \overline{x} it is plain that the Measure of the Velocity of the other Point at s (or the Distance that *might be* uniformly described in the same given Time) will be less than $\overline{x+u} \times \overline{x}$. Which Quantity being, itself, less than $2\overline{x} \times \overline{x}$ (because u is less than x) the Celerity at s must consequently be less than $2\overline{x\overline{x}}$, take the Point s where you will on This-side of S .

Let now r' and s' be any other cōtemporary Positions of the two Points, on the other Side of R and S , and let Ar' be denoted by w : So shall the Distance Ss' , described in the same Time with Rr' ($w-x$), be truly expressed by $ww - xx$, or its Equal $\overline{w+x} \times \overline{w-x}$ (by Hypothesis) Which Distance, as the Velocity of the de-

For, if we look upon them as real Magnitudes, it is objected, that their Ratio will not strictly agree with the Ultimate Ratio assigned: And if, on the other hand, they be taken as mere Nothings, we then lose the very Idea of Proportion. But our Investigation, as is already observed, is not embarrassed with any such Difficulty: For, though the Distance Ss grows, indeed, less and less, continually, and even vanishes when the generating Point n arrives at S ; yet the Velocity of that Point, which is the Quantity in Question, neither vanishes, nor assumes a new Law; but still continues to increase in the same manner as before. — It may, possibly, be objected, that, as the Measure of the said Velocity is, originally, derived by Means of the Distance nS , we cannot retain a clear Idea of it, when that Distance is vanished out of the Equation. But, with equal Reason, it might be urged, that, we can have no just Conception of the Dimensions and true Proportion of a Building, after the Scaffolding by Means of which it was raised, is taken away.

R

scribing

scribing Point n continually increases, must, evidently, be less than That which would uniformly arise from the Celerity at s' , in the same Time. Whence it is also plain that the said Velocity at s' , is to the uniform Velocity of m , in a greater Ratio than That of $\overline{w+x} \times \overline{w-x}$ to $\overline{w-x}$, or of $\overline{w+x} \times \dot{x}$ to \dot{x} . But the Quantity $\overline{w+x} \times \dot{x}$ is, itself, greater than $2x \times \dot{x}$, because w is greater than x ; and so the Measure of the said Velocity at s must consequently be greater than $2x\dot{x}$.

Therefore, since the Velocity increases continually, from C to D; and seeing the Measure Thereof, before the Arrival of the generating Point at S, is everywhere less, and afterwards every-where greater, than $2x\dot{x}$; it is manifest, that, at S, it can be neither lesser nor greater, but must have, or pass thro', the very Value, or Degree, expressed by $2x\dot{x}$. Q. E. I.

If the Line $Am(x)$ be supposed to be generated with an accelerated, or a retarded Motion, instead of an uniform One, it will readily appear, from the first of the foregoing Methods, that the required Fluxion of x^2 , supposing \dot{x} to denote the Measure of the Velocity at R, will, *still*, be expounded by $2x\dot{x}$.

For the Spaces $rR(v)$ and $sS(2xv - vv)$ described in the same Time, being to each other, in the Ratio of \dot{x} to $2x\dot{x} - v\dot{x}$, the Mean Celerities of the generating Motions, at certain intermediate Points between the extreme Ones r, R , and s, S , must be likewise, in that Ratio: Which Ratio, when v becomes $= 0$, and the Points coincide, will become That of \dot{x} to $2x\dot{x}$.

PROPOSITION II.

11. *The Fluxions, \dot{x} and \dot{y} , of two flowing Quantities, x and y , being given; 'tis proposed to determine the Fluxion of the Rectangle, or Product, xy , of the said Quantities.*

Let z be, *always*, equal to the Sum of the two proposed Quantities x and y : Then, the Fluxions of equal Quantities being also equal, \dot{z} must be $= \dot{x} + \dot{y}$. Moreover, since z is $= x + y$, we shall have $z\dot{z} = x\dot{x} + 2xy\dot{z} + y\dot{y}$; and therefore $xy\dot{z} = \frac{1}{2}z\dot{z} - \frac{1}{2}x\dot{x} - \frac{1}{2}y\dot{y}$. But the Fluxion of $\frac{1}{2}z\dot{z} - \frac{1}{2}x\dot{x} - \frac{1}{2}y\dot{y}$ (and consequently That of its Equal $xy\dot{z}$) appears, from the preceding Proposition, to be $\dot{z}z - x\dot{x} - y\dot{y}$: Which, by writing $x + y$, and $\dot{x} + \dot{y}$, in the Room of their Equals z and \dot{z} , will become $(x + y) \times (\dot{x} + \dot{y}) - x\dot{x} - y\dot{y} = y\dot{x} + x\dot{y}$, the required Fluxion of xy . Hence it is apparent that the Fluxion of the Product, or Rectangle, of any two flowing Quantities, is expressed by the Sum of the Products arising from the Multiplication of each Quantity into the Fluxion of the Other.

12. From the Fluxion of a Rectangle, above determined, the Fluxion of a Fraction, $\frac{z}{y}$, is very easily deduced.

For, by putting $x = \frac{z}{y}$ (the proposed Fraction) and multiplying both Sides of the Equation by y we have $xy = z$; and therefore $x\dot{y} + y\dot{x} = \dot{z}$, *as above*, (the Fluxions of equal Quantities being, necessarily, equal). From this Equation, by transposing $x\dot{y}$, and dividing the Whole by y , we get $\dot{x} = \frac{\dot{z}}{y} - \frac{z\dot{y}}{y^2}$: This, by writing $\frac{z}{y}$ in the Room of its Equal x , becomes $\dot{x} = \frac{\dot{z}}{y} -$

$\frac{zy}{yy} = \frac{y\dot{z} - z\dot{y}}{yy}$: Which Value is therefore the true

Fluxion of x , or, its Equal, $\frac{z}{y}$, the Fraction proposed.

13. Moreover, from the Fluxion of a Rectangle, the Fluxion of the continual Product of three, four, five, or any other Number, of flowing Quantities, may be determined.

Thus, let the Fluxion of yzu , where the Number of Factors is 3, be first required: Then, by putting $x = zu$, our given Expression will be reduced to yx ; and its Fluxion will be $y\dot{x} + x\dot{y}$ (*by Prop. 2*). But, x being $= zu$, and therefore $\dot{x} = z\dot{u} + u\dot{z}$ (*by the same*), if these Values be substituted in $y\dot{x} + x\dot{y}$, it will become $y \times z\dot{u} + u\dot{z} + zu\dot{y} = yz\dot{u} + y\dot{z}u + jzu$, the true Fluxion of yzu , required.

Again, if the Fluxion of $yzvw$, where the Number of Factors is four, was to be demanded; then, by making $x = zvw$, the Quantity proposed will be reduced to yx ; and its Fluxion will therefore be expressed by $y\dot{x} + x\dot{y}$; which, because x is $= zvw$, and $\dot{x} = z\dot{v}w + z\dot{v}w + \dot{z}vw$ (*as appears from above*) will be likewise expressed by $y \times z\dot{v}w + z\dot{v}w + \dot{z}vw + zvw\dot{y}$, or $yzv\dot{w} + yz\dot{v}w + y\dot{z}vw + jzvw$.

In the same manner the Fluxion of $yzvws$, will appear to be $yzvws\dot{y} + yz\dot{v}ws + yx\dot{v}ws + y\dot{z}vws + jzvws$; and so of others.

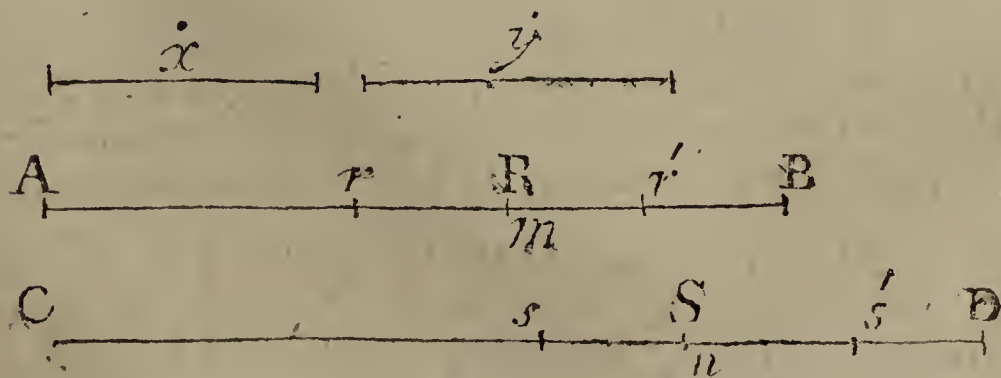
14. From the Fluxions thus determined, the Fluxion of any Power of a variable Quantity (whose Exponent is a whole positive Number) is very readily obtained; nothing more being herein required, than to consider all the Factors as equal among themselves. Thus, the

the Fluxion of yvw being found $y\dot{v}\dot{w} + y\dot{v}w + yv\dot{w}$, it is plain, if both v and w be supposed equal to y , that the Fluent, or Quantity propounded will become y^3 , and its Fluxion $yy\dot{y} + y\dot{y}y + y\dot{y}y = 3y^2\dot{y}$.

Thus, also, because the Fluxion of $yzvw$ is $yz\dot{v}\dot{w} + yz\dot{v}w + yz\dot{v}w + yz\dot{v}w$, it appears that the Fluxion of y^4 will be truly expressed by $4y^3\dot{y}$. And, from the given Fluxion of $yzvws$, That of y^5 will in like manner appear to be $5y^4\dot{y}$. From whence the Law of Continuation is manifest; the Fluxion of y^p being universally expounded by $py^{p-1}\dot{y}$.

15. This last General Conclusion, which is of very great Importance in the Business of Fluxions, being the Result of several Deductions, whereby its Truth and Evidence may, perhaps, lose a Part of their Force, a more direct Investigation of the *same* may not here be amiss; though the former Method will, doubtless, appear the most easy and proper for Beginners, to whom, the manner of working by General Indices, is not plain and familiar.

Conceive two Points m and n to move, at the same Time, from two other Points A and C , along the Right-lines AB and CD ; and let every Thing be supposed as in Prop. 1; *only*, let the Measure of the Distance described by the Point n be, *always*, equal to the p Power, (instead of the Square) of That described by the other Point m , moving uniformly.



R 3

Then,

246 *Some Account of FLUXIONS, with the*

Then, since by Hypothesis, the Value of CS is here $= x^p$, and That of Cs $= u^p$, (*See the second Solution to the foresaid Prop.*) it is evident that the Distance sS, described in the same Time with rR ($= x-u$), will be truly defined by $x^p - u^p$, or its Equal $\frac{x \times x^{p-1} + x^{p-2}u + x^{p-3}u^2 + x^{p-4}u^3 + \dots + u^{p-1}}{x - u}$.

Whence, supposing \dot{x} to denote the Measure of the uniform Velocity of m , it will appear, by Reasoning as in the said Proposition, that the Measure of the Velocity of n , at any Place s , on This-side of S, must be less than $\dot{x} \times \frac{x^{p-1} + x^{p-2}u + x^{p-3}u^2 + \dots + u^{p-1}}{x - u}$, and consequently less than $\dot{x} \times p x^{p-1}$; seeing each of the (p) Terms of the said Series (the First only excepted) is less than x^{p-1} , u being less than x .

Again, by considering r' and s' as two other, cotemporary Positions, beyond R and S, and Reasoning in the same manner, the Measure of the Velocity at s' will appear to be greater, now, than the abovesaid Quantity $\dot{x} \times p x^{p-1}$.

Therefore, as the Velocity increases continually, and seeing the Value thereof, before the Point arrives at S, is every-where less; and afterwards, every-where greater, than $\dot{x} \times p x^{p-1}$ (or $p x^{p-1} \dot{x}$) it is evident, that, at S, it must be neither lesser nor greater, but exactly equal to $p x^{p-1} \dot{x}$: Which Quantity is therefore the true Fluxion of x^p ; and agrees exactly with That determined above.

16. In bringing out the foregoing Fluxion of x^p , the Value of p , in both Methods, was taken as a Whole Number: Nevertheless the Conclusion itself holds equally true, when p is a Fraction, as in the Notation of Roots.

To make this appear, let the Quantity $x^{\frac{3}{2}}$ ($= \sqrt{x^3}$) be propounded, in order to determine the Fluxion Thereof: Which, by writing $\frac{3}{2}$ for p , in the General Fluxion

Fluxion $p x^{p-1} \dot{x}$, comes out $\frac{3}{2} x^{\frac{1}{2}} \dot{x}$. Now, to prove that this is the true Fluxion, put $y = x^{\frac{3}{2}}$, the Quantity given; and then, by squaring both Sides, you will have $y^2 = x^3$; which, in Fluxions, gives $2 y \dot{y} = 3 x^2 \dot{x}$ (as has been already shewn). This, by substituting for y , becomes $2 x^{\frac{3}{2}} \dot{y} = 3 x^2 \dot{x}$. Whence, by Division, $\dot{y} = \frac{3}{2} x^{\frac{1}{2}} \dot{x}$, the very same as before.

But, to demonstrate the same Thing in a General Manner, suppose the Fluxion of $x^{\frac{m}{n}}$ to be required (m and n being any whole Numbers, whatever) Put $y = x^{\frac{m}{n}}$; and then, by raising both Sides to the Power n , you will have $y^n = x^m$: Which, in Fluxions, gives $n y^{n-1} \dot{y} = m x^{m-1} \dot{x}$; and, consequently, $\dot{y} = \frac{m}{n} \times \frac{x^{m-1} \dot{x}}{y^{n-1}} = \frac{m}{n} \times$

$$\frac{y x^{m-1} \dot{x}}{y^n} = \frac{m}{n} \times \frac{y x^{m-1} \dot{x}}{x^m} = \frac{m}{n} \times \frac{x^{\frac{m}{n}} x^{m-1} \dot{x}}{x^m} = \frac{m}{n} \times x^{\frac{m}{n}-1} \dot{x}.$$

Which Value, of the Fluxion of $x^{\frac{m}{n}}$, is evidently the very same, as That arising by expounding p in the General Fluxion ($p x^{p-1} \dot{x}$) by $\frac{m}{n}$; which was to be proved.

17. Now, from what has been thus far delivered, the following practical Rules, for determining the Fluxions of Algebraic Quantities, are obtained.

R U L E I.

To find the Fluxion of any given Power of a flowing Quantity.

Multiply the Fluxion of the Root by the Exponent of the given Power, and the Product into that Power of the same

R 4

Root

248 *Some Account of* FLUXIONS, *with the*

Root which arises by subtracting Unity from the given Exponent.

The Reason of this Rule is seen above; the Rule itself being nothing more than $p \propto x^{p-1}$ (the Fluxion of x^p) expressed in Words.

R U L E II.

To find the Fluxion of the Product of several variable Quantities, multiplied together.

Multiply the Fluxion of each, by the Product of the rest of the Quantities; so shall the Sum of all the Products thus arising be the true Fluxion required.

The Reason of which is, likewise, evident from what has been already delivered. See Art. 13.

R U L E III.

To find the Fluxion of a Fraction, arising from the Division of one variable Quantity by another.

From the Fluxion of the Numerator, drawn into the Denominator, subtract the Fluxion of the Denominator, drawn into the Numerator; and divide the Remainder by the Square of the Denominator.

This appears from $\frac{y\dot{z} - z\dot{y}}{yy}$, the Fluxion of $\frac{z}{y}$, determined in Art. 12.

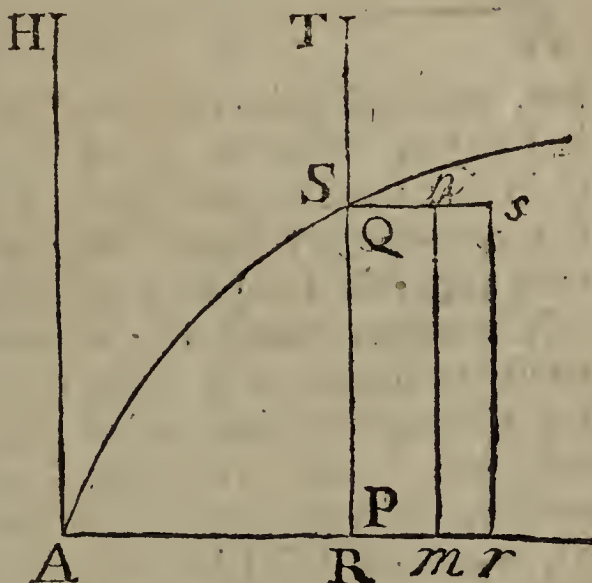
Though I might here, with Propriety enough, put an End to this Part, as my professed Design therein extends no farther than Giving the Young Beginner some Account of the Nature, and First Principles, of Fluxions, together with the Investigation of the Fundamental Rules exhibited above; nevertheless, as different Ways of bringing out the same Truths have often a very good Effect, and seeing the Fluxions of all Quantities whatever (whether Powers, Fractions, &c.) are deducible from the Fluxion of a Rectangle, I shall, therefore, subjoin a different Method, whereby the said Fluxion

Fluxion of a Rectangle (*given by Prop. 2.*) may be investigated: In order to which it will be requisite, first of all, to premise the following

L E M M A.

18. *The Fluxion of a curvilineal Space ARS, generated by the Ordinate RS (or the intercepted Part of a Right-line RT (moving parallel to itself, is equal to the Rectangle (Rs) under the said Ordinate, and the Fluxion (Rr) of the Abscissa AR.*

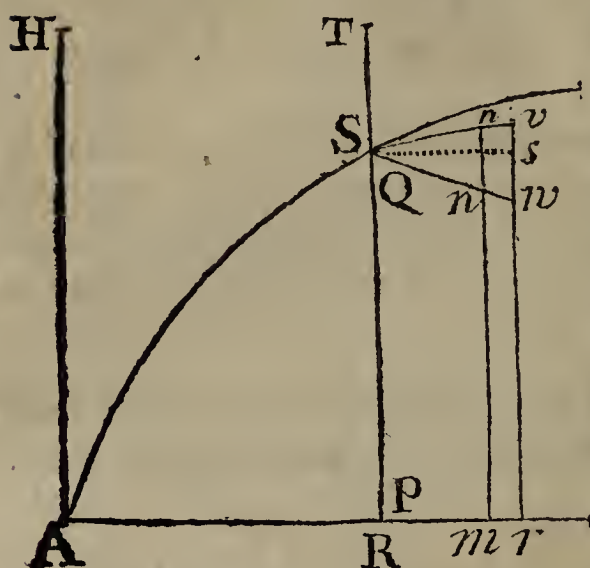
For, let a Right-line mn , of the same Length with RS in the proposed Position PQ , be conceived to move from thence, parallel to itself, with the same Celerity that the generating Line itself has in that Position: By which Means the Rectangle $PrsQ$ will be *uniformly* generated, with the very Celerity by which it begins to be generated, or, by which the Space ARS is increased in the proposed Position PQ ; since both the Length, and Velocity of mn , are the same as Those of RS in the said Position. Hence the Rectangle so generated must, consequently, be the true Fluxion of the Space ARS , by the Definition.



But the same Thing may be otherwise made to appear, from a different, and more logical Method of Arguing; by proving that the required Fluxion can neither be greater, nor less, than the said Rectangle.

Thus,

Thus, if the Line mn (while it moves uniformly on towards rs) be supposed to increase in Length, the Area $PmnQ$, generated thereby, will evidently be greater



than That which would uniformly arise in the same Time, with the given Length at the first Position PQ ; since the new Parts, produced each succeeding Moment (as the generating Line continues to lengthen) are greater and greater.

And, in the same manner, if the Line

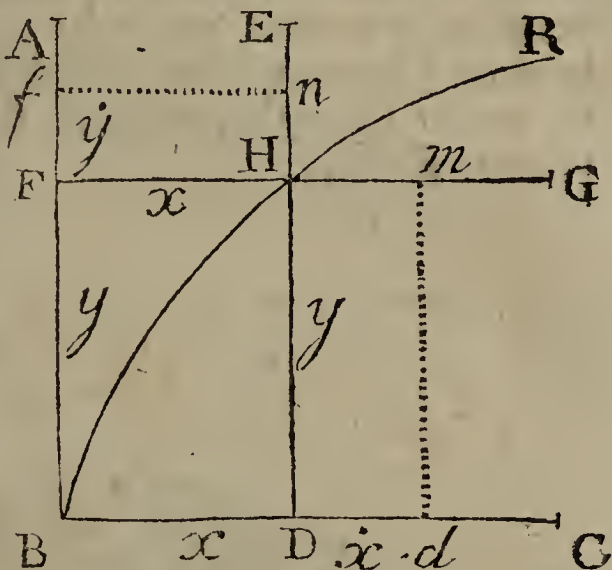
mn , generating the Fluxion, be supposed to decrease, in Length, from the given Position PQ , it is equally plain that the generated Space $PmnQ$ will be less than the cotemporary Space that would, uniformly, arise with the given Length at the first Position PQ .

Therefore, seeing the Fluxion (or the Space that would uniformly arise from the generating Celerity at the proposed Position) is less than any Space that can be described, in the given Time, when the Line mn increases, and greater than any Space that can be described, when the said Line decreases; it must consequently be equal to that Space which will arise, when the Length of the said Line, from the given Position, is supposed neither to increase nor decrease; that is, when the generated Space $PmnQ$ is a Rectangle, as in the preceding Figure.

PROPOSITION.

19. *To determine the Fluxion of the Product, or Rectangle, of two variable Quantities (x and y).*

Let two Right-lines DE and FG, perpendicular to each other, be conceived to move from two other Right-lines BA and BC, continually parallel to themselves, and thereby generate the variable Rectangle DF: Let the Path of their Intersection, or the Place of the Angle H, be the Line BHR, dividing the generated Rectangle DF in two Parts, BDH and BHF: Moreover let $Dd(\dot{x})$ and $Ff(\dot{y})$ be the Fluxions of the Sides BD (x) and BF (y); and suppose dm and fn to be drawn parallel, and equal, to DH and FH, respectively.



Then, since, by the preceding Lemma, the Fluxion of the Space or Area BDH, is truly expressed by the Rectangle $Dm (=y\dot{x})$ and That of the Space or Area BHF, by the Rectangle $Fn (=x\dot{y})$, it follows, because equal Quantities have equal Fluxions, that the Fluxion of the proposed Rectangle $xy (=BDH + BHF)$ is truly expressed by $x\dot{y} + y\dot{x}$, the very Expression before determined: See Art. 11.

It may, perhaps, be expected, that I should now give some Instances of the Use and Application of the Theory hitherto explained, in the Resolution of Problems: But, having insisted very largely on this Head in my *Doctrine and Application of Fluxions*, I shall take the Liberty to recommend that Work, to the Perusal of Such as are desirous of farther Information in the Matter. Those, for whose Use the above Account is in a more particular Manner designed, may have the Opportunity of being instructed in the Practice, by proper Examples, without the Trouble of turning to other Books.





P A R T VI.

TREATING OF

The VALUATION of ANNUITIES, for single and joint LIVES.

THE following Tract being designed for *real Use*, and adapted to the Understanding of Such whose Acquisitions, in the Mathematical Way, extend not beyond Vulgar, or Decimal Arithmetick; it therefore seem'd proper to omit the Investigation of several Particulars therein delivered, depending upon higher Principles. This, I hope, my Mathematical Readers will have the Candour to excuse; when They consider the Importance of the Subject to a Multitude of Persons, who cannot be expected to see into the Nature and Usefulness of an Algebraical Process.—Without further Apology I shall therefore proceed now to my Purpose; which is to exhibit, in a plain, easy manner, by Means of proper Tables, the practical Solutions of the most useful and necessary Problems on the Subject; without any Intermixture of *Analytical Operations* (whereof the bare Appearance, to Those unacquainted therewith, would seem to cast a Darkness over the *Whole*) However, for the sake of Those that are Judges, I shall, in an annex'd Scholium, give the Reasons of what is most material, and necessary to be explained.

TABLE

T A B L E I.

*Shewing the Probability of the Duration of Life,
from Observations on the Bills of Mortality of the
City of London.*

Years.	N ^o . Perfons.	Years.	N ^o . Perfons.	Years.	N ^o . Perfons.	Years.	N ^o . Perfons.
0	1000	20	360	40	229	60	102
1	680	21	355	41	222	61	97
2	547	22	350	42	214	62	92
3	496	23	345	43	206	63	87
4	469	24	339	44	199	64	82
5	452	25	333	45	192	65	77
6	440	26	327	46	185	66	72
7	430	27	321	47	178	67	67
8	422	28	315	48	171	68	62
9	415	29	308	49	165	69	58
10	410	30	301	50	159	70	54
11	405	31	294	51	153	71	50
12	400	32	287	52	147	72	46
13	395	33	280	53	141	73	42
14	390	34	273	54	135	74	39
15	385	35	266	55	129	75	36
16	380	36	259	56	123	76	33
17	375	37	252	57	117	77	30
18	370	38	245	58	112	78	27
19	365	39	237	59	107	79	25

T A B L E II.

Exhibiting the Number of Years of Life, which a Person, of a given Age, may, upon an Equality of Chance, expect to enjoy; according to the aforesaid Observations.

Age.	Exp.	Age.	Exp.	Age.	Exp.	Age.	Exp.
1	27.0	21	28.3	41	19.2	61	12.0
2	32.0	22	27.7	42	18.8	62	11.6
3	34.0	23	27.2	43	18.5	63	11.2
4	35.6	24	26.6	44	18.1	64	10.8
5	36.0	25	26.1	45	17.8	65	10.5
6	36.0	26	25.6	46	17.4	66	10.1
7	35.8	27	25.1	47	17.0	67	9.8
8	35.6	28	24.6	48	16.7	68	9.4
9	35.2	29	24.1	49	16.3	69	9.1
10	34.8	30	23.6	50	16.0	70	8.8
11	34.3	31	23.1	51	15.6	71	8.4
12	33.7	32	22.7	52	15.2	72	8.1
13	33.1	33	22.3	53	14.9	73	7.8
14	32.5	34	21.9	54	14.5	74	7.5
15	31.9	35	21.5	55	14.2	75	7.2
16	31.3	36	21.1	56	13.8	76	6.8
17	30.7	37	20.7	57	13.4	77	6.4
18	30.1	38	20.3	58	13.1	78	6.0
19	29.5	39	19.9	59	12.7	79	5.5
20	28.9	40	19.6	60	12.4	80	5.0

T A B L E III.

A TABLE shewing the present Value of one Pound, to be received at the end of any number of Years, not exceeding 90, discounting at the Rates of 5, 4, and 3 per Cent. compound Interest.

Years.	Value at 5 per cent.	Value at 4 per cent.	Value at 3 per cent.	Years.	Value at 5 per cent.	Value at 4 per cent.	Value at 3 per cent.
1	.9524	.9615	.9709	21	.3589	.4388	.5375
2	.9070	.9245	.9426	22	.3418	.4219	.5219
3	.8638	.8890	.9151	23	.3255	.4057	.5067
4	.8227	.8548	.8885	24	.3100	.3901	.4919
5	.7835	.8219	.8626	25	.2953	.3757	.4776
6	.7462	.7903	.8375	26	.2812	.3607	.4637
7	.7107	.7599	.8131	27	.2678	.3468	.4502
8	.6768	.7307	.7894	28	.2551	.3335	.4371
9	.6446	.7026	.7664	29	.2429	.3206	.4243
10	.6139	.6756	.7441	30	.2314	.3083	.4120
11	.5847	.6496	.7224	31	.2204	.2965	.4000
12	.5568	.6246	.7014	32	.2099	.2851	.3883
13	.5303	.6006	.6809	33	.1999	.2741	.3770
14	.5051	.5775	.6611	34	.1903	.2636	.3660
15	.4810	.5553	.6419	35	.1813	.2534	.3554
16	.4581	.5339	.6232	36	.1726	.2437	.3450
17	.4363	.5134	.6050	37	.1644	.2343	.3350
18	.4155	.4936	.5874	38	.1566	.2253	.3252
19	.3957	.4746	.5703	39	.1491	.2166	.3158
20	.3769	.4564	.5537	40	.1420	.2083	.3066

Years.	Value at 5 per cent.	Value at 4 per cent.	Value at 3 per cent.	Years.	Value at 5 per cent.	Value at 4 per cent.	Value at 3 per cent.
41	.1353	.2003	.2977	66	.0399	.0751	.1421
42	.1288	.1926	.2890	67	.0380	.0722	.1380
43	.1227	.1852	.2805	68	.0362	.0695	.1340
44	.1169	.1780	.2724	69	.0345	.0668	.1301
45	.1113	.1712	.2644	70	.0329	.0642	.1263
46	.1060	.1646	.2567	71	.0313	.0617	.1226
47	.1010	.1583	.2493	72	.0298	.0594	.1190
48	.0961	.1522	.2420	73	.0284	.0571	.1156
49	.0916	.1463	.2349	74	.0270	.0549	.1122
50	.0872	.1407	.2281	75	.0257	.0528	.1089
51	.0830	.1353	.2215	76	.0245	.0508	.1058
52	.0791	.1301	.2150	77	.0233	.0488	.1027
53	.0753	.1251	.2087	78	.0222	.0469	.0997
54	.0717	.1203	.2027	79	.0212	.0451	.0968
55	.0683	.1156	.1968	80	.0202	.0434	.0940
56	.0651	.1112	.1910	81	.0192	.0417	.0912
57	.0620	.1069	.1855	82	.0183	.0401	.0886
58	.0590	.1028	.1801	83	.0174	.0386	.0860
59	.0562	.0989	.1748	84	.0166	.0371	.0835
60	.0535	.0951	.1697	85	.0158	.0357	.0811
61	.0510	.0914	.1648	86	.0151	.0343	.0787
62	.0485	.0879	.1600	87	.0143	.0330	.0764
63	.0462	.0845	.1553	88	.0136	.0317	.0742
64	.0440	.0813	.1508	89	.0130	.0305	.0720
65	.0419	.0781	.1464	90	.0124	.0292	.0699

T A B L E IV.

A TABLE shewing the present Value of an Annuity of one Pound for any number of Years, not exceeding 90, when Interest is at 5, 4, and 3 per Cent.

Years.	Value at 5 per cent.	Value at 4 per cent.	Value at 3 per cent.	Years.	Value at 5 per cent.	Value at 4 per cent.	Value at 3 per cent.
1	0.952	0.961	0.971	21	12.821	14.029	15.415
2	1.859	1.886	1.913	22	13.163	14.451	15.939
3	2.723	2.775	2.829	23	13.488	14.857	16.444
4	3.546	3.630	3.717	24	13.799	15.247	16.936
5	4.329	4.452	4.580	25	14.094	15.622	17.413
6	5.076	5.242	5.497	26	14.375	15.983	17.877
7	5.786	6.002	6.230	27	14.643	16.329	18.327
8	6.463	6.733	7.020	28	14.898	16.663	18.764
9	7.108	7.435	7.786	29	15.141	16.984	19.188
10	7.721	8.111	8.530	30	15.372	17.292	19.600
11	8.306	8.760	9.253	31	15.593	17.588	20.000
12	8.863	9.385	9.954	32	15.803	17.873	20.389
13	9.393	9.985	10.635	33	16.002	18.148	20.766
14	9.899	10.563	11.296	34	16.193	18.411	21.132
15	10.380	11.118	11.938	35	16.374	18.665	21.487
16	10.838	11.652	12.561	36	16.547	18.908	21.832
17	11.274	12.166	13.166	37	16.711	19.142	22.167
18	11.689	12.659	13.753	38	16.868	19.368	22.492
19	12.085	13.134	14.324	39	17.017	19.584	22.808
20	12.462	13.590	14.877	40	17.159	19.793	23.115

Years.	Value at 5 per cent.	Value at 4 per cent.	Value at 3 per cent.	Years.	Value at 5 per cent.	Value at 4 per cent.	Value at 3 per cent.
41	17.294	19.993	23.412	66	19.201	23.122	28.595
42	17.423	20.186	23.701	67	19.239	23.194	28.733
43	17.546	20.371	23.982	68	19.275	23.263	28.867
44	17.663	20.549	24.254	69	19.310	23.330	28.997
45	17.774	20.720	24.519	70	19.343	23.394	29.123
46	17.880	20.885	24.775	71	19.374	23.456	29.246
47	17.981	21.043	25.025	72	19.404	23.516	29.365
48	18.077	21.195	25.267	73	19.432	23.573	29.481
49	18.169	21.341	25.502	74	19.459	23.628	29.593
50	18.256	21.482	25.730	75	19.485	23.680	29.702
51	18.339	21.617	25.951	76	19.509	23.731	29.808
52	18.418	21.747	26.166	77	19.533	23.780	29.910
53	18.493	21.873	26.375	78	19.555	23.827	30.010
54	18.565	21.993	26.578	79	19.576	23.872	30.108
55	18.633	22.109	26.774	80	19.596	23.915	30.201
56	18.698	22.220	26.965	81	19.615	23.957	30.292
57	18.760	22.327	27.151	82	19.634	23.997	30.381
58	18.819	22.430	27.331	83	19.652	24.036	30.467
59	18.876	22.528	27.506	84	19.668	24.073	30.550
60	18.929	22.623	27.676	85	19.684	24.108	30.631
61	18.980	22.715	27.840	86	19.699	24.143	30.710
62	19.029	22.803	28.000	87	19.713	24.176	30.786
63	19.075	22.887	28.156	88	19.727	24.207	30.860
64	19.119	22.968	28.306	89	19.740	24.238	30.932
65	19.161	23.046	28.453	90	19.752	24.267	31.002

T A B L E V.

*For the Valuation of Annuities upon one
L I F E.*

Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.	Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.
6	14.1	16.2	18.8	21	12.9	14.7	17.0
7	14.2	16.3	18.9	22	12.7	14.5	16.8
8	14.3	16.4	19.0	23	12.6	14.3	16.5
9	14.3	16.4	19.0	24	12.4	14.1	16.3
10	14.3	16.4	19.0	25	12.3	14.0	16.1
11	14.3	16.4	19.0	26	12.1	13.8	15.9
12	14.2	16.3	18.9	27	12.0	13.6	15.6
13	14.1	16.2	18.7	28	11.8	13.4	15.4
14	14.0	16.0	18.5	29	11.7	13.2	15.2
15	13.9	15.8	18.3	30	11.6	13.1	15.0
16	13.7	15.6	18.1	31	11.4	12.9	14.8
17	13.5	15.4	17.9	32	11.3	12.7	14.6
18	13.4	15.2	17.6	33	11.2	12.6	14.4
19	13.2	15.0	17.4	34	11.0	12.4	14.2
20	13.0	14.8	17.2	35	10.9	12.3	14.1

Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.	Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.
36	10.8	12.1	13.9	56	8.4	9.1	10.1
37	10.6	11.9	13.7	57	8.2	8.9	9.9
38	10.5	11.8	13.5	58	8.1	8.7	9.6
39	10.4	11.6	13.3	59	8.0	8.6	9.4
40	10.3	11.5	13.2	60	7.9	8.4	9.2
41	10.2	11.4	13.0	61	7.7	8.2	8.9
42	10.1	11.2	12.8	62	7.6	8.1	8.7
43	10.0	11.1	12.6	63	7.4	7.9	8.5
44	9.9	11.0	12.5	64	7.3	7.7	8.3
45	9.8	10.8	12.3	65	7.1	7.5	8.0
46	9.7	10.7	12.1	66	6.9	7.3	7.8
47	9.5	10.5	11.9	67	6.7	7.1	7.6
48	9.4	10.4	11.8	68	6.6	6.9	7.4
49	9.3	10.2	11.6	69	6.4	6.7	7.1
50	9.2	10.1	11.4	70	6.2	6.5	6.9
51	9.0	9.9	11.2	71	6.0	6.3	6.7
52	8.9	9.8	11.0	72	5.8	6.1	6.5
53	8.8	9.6	10.7	73	5.6	5.9	6.2
54	8.6	9.4	10.5	74	5.4	5.6	5.9
55	8.5	9.3	10.3	75	5.2	5.4	5.6

T A B L E VI.

Exhibiting the Value of an Annuity for a given Term of Years, on the Contingency of its ceasing upon the Extinction of an assigned Life.

Given Age.	N. Years.	Val. at 5 per Cent.	Val. at 4 p. Cent.	Val. at 3 per Cent.	Given Age.	N. Years.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.
5	5	4.0	4.1	4.2	15	5	4.1	4.2	4.4
	10	7.0	7.3	7.7		10	7.1	7.5	7.9
	15	9.1	9.8	10.5		15	9.3	10.0	10.7
	20	10.7	11.7	12.8		20	10.8	11.8	12.9
	25	11.8	13.0	14.5		25	11.8	13.1	14.5
	30	12.6	14.1	15.8		30	12.5	14.0	15.7
	35	13.2	14.9	16.8		35	13.0	14.6	16.6
	40	13.6	15.4	17.5		40	13.3	15.0	17.2
	45	13.8	15.7	18.0		45	13.5	15.3	17.6
	50	13.9	15.9	18.3		50	13.6	15.5	17.9
10	5	4.1	4.2	4.3	20	5	4.1	4.2	4.3
	10	7.2	7.6	8.0		10	7.0	7.4	7.8
	15	9.4	10.1	10.8		15	9.1	9.7	10.5
	20	11.0	12.0	13.1		20	10.5	11.4	12.5
	25	12.1	13.4	14.8		25	11.4	12.6	13.9
	30	12.8	14.4	16.2		30	12.0	13.4	15.0
	35	13.3	15.1	17.1		35	12.5	13.9	15.7
	40	13.7	15.5	17.7		40	12.7	14.3	16.3
	45	13.9	15.8	18.2		45	12.8	14.5	16.6
	50	14.0	16.0	18.5		50	12.9	14.6	16.8

Given Age.	N. Years.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.	Given Age.	N. Years.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.
25	5	4.1	4.2	4.3	45	5	3.9	4.0	4.1
	10	6.9	7.3	7.6		10	6.3	6.7	7.0
	15	8.9	9.5	10.1		15	7.9	8.4	9.0
	20	10.2	11.0	12.0		20	8.8	9.5	10.3
	25	11.0	12.1	13.4		25	9.4	10.2	11.2
	30	11.5	12.8	14.3		30	9.6	10.6	11.7
	35	11.8	13.2	15.0	55	5	3.8	3.9	4.0
	40	12.0	13.5	15.4		10	6.2	6.5	6.8
30	45	12.1	13.7	15.6		15	7.6	8.2	8.8
	5	4.0	4.1	4.2		20	8.5	9.1	9.9
	10	6.8	7.1	7.5		25	8.9	9.7	10.6
	15	8.6	9.2	9.8		5	3.8	3.9	4.0
	20	9.8	10.6	11.5		10	6.1	6.4	6.7
	25	10.6	11.6	12.7		15	7.4	7.9	8.4
	30	11.0	12.2	13.5		20	8.0	8.7	9.4
	35	11.2	12.6	14.0		25	8.3	9.0	9.9
35	40	11.4	12.8	14.4	60	5	3.7	3.8	3.9
	5	4.0	4.1	4.2		10	5.8	6.1	6.4
	10	6.6	6.9	7.3		15	7.0	7.4	7.9
	15	8.3	8.9	9.5		20	7.5	8.0	8.7
	20	9.4	10.2	11.1		5	3.6	3.7	3.8
	25	10.1	11.1	12.2		10	5.4	5.7	6.0
	30	10.5	11.6	13.0		15	6.4	6.8	7.2
	35	10.7	11.9	13.5	70	5	3.4	3.5	3.6
40	5	3.9	4.0	4.1		10	5.0	5.2	5.4
	10	6.4	6.8	7.1					
	15	8.0	8.6	9.2					
	20	9.1	9.8	10.6					
	25	9.7	10.6	11.6					
	30	10.0	11.0	12.2					

T A B L E VII.

Serving as a Supplement to TABLE VI.

Diff.	1	2	3	4
3.4	0.8	1.6	2.3	2.9
3.3	0.8	1.5	2.2	2.8
3.2	0.8	1.5	2.1	2.7
3.1	0.7	1.4	2.0	2.6
3.0	0.7	1.3	1.9	2.5
2.9	0.7	1.3	1.9	2.4
2.8	0.7	1.3	1.8	2.3
2.7	0.6	1.2	1.7	2.2
2.6	0.6	1.2	1.7	2.2
2.5	0.6	1.1	1.6	2.1
2.4	0.6	1.1	1.6	2.0
2.3	0.5	1.0	1.5	1.9
2.2	0.5	1.0	1.4	1.8
2.1	0.5	0.9	1.3	1.7
2.0	0.5	0.9	1.3	1.7
1.9	0.4	0.8	1.2	1.6
1.8	0.4	0.8	1.1	1.5
1.7	0.4	0.7	1.1	1.4
1.6	0.4	0.7	1.0	1.3
1.5	0.4	0.7	1.0	1.3
1.4	0.3	0.6	0.9	1.2
1.3	0.3	0.6	0.9	1.1
1.2	0.3	0.6	0.8	1.0
1.1	0.3	0.5	0.7	0.9
1.0	0.2	0.4	0.6	0.8

TABLE

1872-1873

1872-1873

3

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T A B L E VIII.

*Shewing the Value of an Annuity for two joint Lives
(i. e. for as long as They exist together).*

Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.	Age of the Youngest	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.
10	10	11.6	13.0	14.7	20	20	10.1	11.3	12.8
	15	11.3	12.7	14.3		25	9.7	10.8	12.2
	20	10.8	12.2	13.8		30	9.2	10.3	11.6
	25	10.2	11.6	13.1		35	8.8	9.8	10.9
	30	9.7	10.9	12.3		40	8.4	9.2	10.2
	35	9.1	10.2	11.5		45	7.9	8.6	9.5
	40	8.6	9.6	10.7		50	7.4	8.0	8.8
	45	8.1	9.0	10.0		55	6.9	7.5	8.1
	50	7.6	8.4	9.3		60	6.4	6.9	7.4
	55	7.1	7.8	8.6		65	5.9	6.3	6.7
	60	6.6	7.2	7.8		70	5.4	5.7	6.0
	65	6.1	6.5	6.9		75	4.8	5.0	5.2
	70	5.5	5.8	6.1	25	25	9.4	10.5	11.8
	75	4.9	5.1	5.3		30	9.0	10.1	11.3
15	15	11.0	12.3	13.9		35	8.6	9.6	10.7
	20	10.5	11.8	13.3		40	8.2	9.1	10.0
	25	10.1	11.2	12.6		45	7.8	8.5	9.4
	30	9.5	10.6	11.9		50	7.3	7.9	8.7
	35	9.0	10.0	11.2		55	6.8	7.4	8.0
	40	8.5	9.4	10.4		60	6.3	6.8	7.3
	45	8.0	8.8	9.6		65	5.8	6.2	6.6
	50	7.5	8.2	8.9		70	5.3	5.6	5.9
	55	7.0	7.6	8.2		75	4.7	4.9	5.1
	60	6.5	7.0	7.5	30	30	8.6	9.6	10.8
	65	6.0	6.4	6.8		35	8.3	9.2	10.3
	70	5.4	5.7	6.0		40	8.0	8.8	9.7
	75	4.8	5.0	5.2					

Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.	Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.
30	45	7.6	8.3	9.1	45	65	5.4	5.8	6.3
	50	7.2	7.8	8.5		70	5.0	5.3	5.6
						75	4.5	4.7	4.9
	55	6.7	7.3	7.9	50	50	6.2	6.8	7.6
	60	6.2	6.7	7.2		55	6.0	6.5	7.2
	65	5.7	6.1	6.5		60	5.7	6.1	6.7
	70	5.2	5.5	5.8		65	5.3	5.7	6.2
	75	4.7	4.9	5.1		70	4.9	5.2	5.5
						75	4.4	4.6	4.8
	35	8.0	8.8	9.9		55	5.7	6.2	6.9
35	40	7.7	8.5	9.4		60	5.5	5.9	6.5
	45	7.4	8.1	8.9		65	5.2	5.6	6.0
	50	7.0	7.6	8.3		70	4.8	5.1	5.4
	55	6.6	7.1	7.7		75	4.3	4.5	4.7
	60	6.1	6.5	7.1	60	60	5.2	5.6	6.1
	65	5.6	6.0	6.4		65	4.9	5.3	5.7
	70	5.1	5.4	5.7		70	4.6	4.9	5.2
	75	4.6	4.8	5.0	65	75	4.2	4.4	4.6
						65	4.7	5.0	5.4
	40	7.3	8.1	9.1		70	4.4	4.6	4.9
40	45	7.1	7.8	8.7	70	75	3.9	4.0	4.2
	50	6.8	7.4	8.2		75	3.6	3.7	3.8
	55	6.4	6.9	7.6					
	60	6.0	6.4	7.0	75	75	3.6	3.7	3.8
	65	5.5	5.9	6.4					
	70	5.1	5.4	5.7					
	75	4.6	4.8	5.0					
	45	6.7	7.4	8.3					
	50	6.5	7.1	7.9					
45	55	6.2	6.7	7.4					
	60	5.8	6.3	6.8					

TABLE

T A B L E IX.

For the Value of an Annuity upon the longest of two given Lives.

Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.	Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.
10	10	17.1	19.9	23.4	20	20	15.8	18.3	21.6
	15	16.8	19.5	22.9		25	15.5	17.9	21.1
	20	16.6	19.1	22.5		30	15.3	17.6	20.7
	25	16.4	18.8	22.2		35	15.1	17.4	20.4
	30	16.2	18.6	21.9		40	15.0	17.2	20.1
	35	16.1	18.4	21.6		45	14.9	17.0	19.9
	40	16.0	18.3	21.4		50	14.7	16.8	19.6
	45	15.9	18.2	21.2		55	14.5	16.6	19.4
	50	15.8	18.0	20.9		60	14.3	16.3	19.1
	55	15.7	17.8	20.7		65	14.1	16.0	18.7
	60	15.5	17.6	20.4		70	13.8	15.7	18.2
	65	15.3	17.4	20.1		75	13.5	15.3	17.7
	70	15.1	17.2	19.8	25	25	15.1	17.4	20.3
	75	14.8	16.9	19.5		30	14.9	17.0	19.8
15	15	16.7	19.3	22.8		35	14.7	16.7	19.4
	20	16.4	18.9	22.3		40	14.5	16.5	19.2
	25	19.2	18.6	21.9		45	14.3	16.3	18.9
	30	16.0	18.3	21.6		50	14.2	16.1	18.7
	35	15.9	18.1	21.3		55	14.0	15.9	18.4
	40	15.7	17.9	21.1		60	13.8	15.6	18.0
	45	15.6	17.8	20.9		65	13.6	15.3	17.6
	50	15.4	17.6	20.7		70	13.3	15.0	17.2
	55	15.3	17.4	20.4		75	12.9	14.6	16.7
	60	15.2	17.2	20.1	30	30	14.5	16.6	19.3
	65	15.0	16.9	19.8		35	14.2	16.2	18.8
	70	14.7	16.6	19.4		40	14.0	15.9	18.4
	75	14.4	16.3	18.9		45	13.8	15.6	18.1

Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.	Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.
30	50	13.6	15.4	17.8	45	65	11.4	12.5	14.1
	55	13.4	15.1	17.4		70	11.0	12.0	13.6
	60	13.2	14.8	17.0		75	10.6	11.6	13.1
	65	12.9	14.5	16.6					
	70	12.6	14.1	16.1	50	50	12.1	13.3	15.0
	75	12.2	13.7	15.6		55	11.7	12.9	14.5
35						60	11.3	12.4	13.9
	35	13.8	15.8	18.3		65	10.9	12.0	13.3
	40	13.5	15.4	17.8		70	10.5	11.5	12.8
	45	13.3	15.1	17.4		75	10.1	11.0	12.3
	50	13.1	14.8	17.1					
	55	12.9	14.5	16.7		55	11.3	12.4	13.6
	60	12.7	14.2	16.3		60	10.9	11.9	13.0
	65	12.4	13.8	15.8		65	10.5	11.3	12.4
	70	12.0	13.4	15.3		70	10.0	10.8	11.8
	75	11.6	13.0	14.8		75	9.5	10.3	11.3
40					60	60	10.5	11.2	12.2
	40	13.3	15.0	17.3		65	10.0	10.6	11.5
	45	13.0	14.6	16.8		70	9.5	10.1	10.9
	50	12.7	14.2	16.3		75	9.0	9.5	10.3
	55	12.4	13.9	15.9					
	60	12.1	13.5	15.4	65	65	9.4	10.0	10.7
	65	11.8	13.1	14.9		70	8.9	9.4	10.0
	70	11.4	12.7	14.5		75	8.3	8.7	9.3
	75	11.0	12.3	14.0					
					70	70	8.2	8.6	9.2
45	45	12.8	14.2	16.2		75	7.6	7.9	8.4
	50	12.5	13.8	15.7	75	75	6.9	7.2	7.6
	55	12.1	13.4	15.2					
	60	11.7	12.9	14.7					

TABLE

X
T A B L E X.

For finding the Value of an Annuity upon One, or more Lives, at the Rates of $3\frac{1}{4}$, $3\frac{1}{2}$, $3\frac{3}{4}$, $4\frac{1}{4}$, $4\frac{1}{2}$, and $4\frac{3}{4}$, per Cent. Interest; supposing the Value, at the Rate of 4 per Cent. to be known, from the preceding Tables.

Val. at 4 p. cent	$3\frac{1}{4}$	$3\frac{1}{2}$	$3\frac{3}{4}$	$4\frac{1}{4}$	$4\frac{1}{2}$	$4\frac{3}{4}$
	add	add	add	sub.	sub.	sub.
6	0.3	0.2	0.1	0.1	0.2	0.3
7	0.4	0.2	0.1	0.1	0.2	0.3
8	0.5	0.3	0.1	0.1	0.3	0.4
9	0.7	0.4	0.2	0.2	0.4	0.6
10	0.9	0.6	0.3	0.3	0.5	0.7
11	1.1	0.7	0.3	0.3	0.6	0.9
12	1.3	0.8	0.4	0.4	0.7	1.0
13	1.4	0.9	0.4	0.4	0.8	1.2
14	1.6	1.0	0.5	0.5	0.9	1.3
15	1.7	1.1	0.5	0.5	1.0	1.4
16	1.9	1.3	0.6	0.6	1.1	1.6
17	2.1	1.4	0.7	0.6	1.2	1.8
18	2.3	1.5	0.7	0.7	1.3	1.9
19	2.5	1.6	0.8	0.7	1.4	2.0
20	2.7	1.7	0.8	0.8	1.5	2.2

PROBLEM

PROBLEM I.

To find the present Value of any Sum of Money, to be received at the End of a given Term of Years; discounting at the Rate of 3, 4, or 5 per Cent. compound Interest.

SOLUTION.

Find, by *Tab. III.* the present Value of 1*l.* to be received at the End of the given Term; which multiply by the Number of Pounds proposed (cutting off 4 Figures from the Product, on Account of the Decimals) then the Result will be the Value sought.

EXAMPLE. Let the Sum proposed be 800*l.* the given Term 7 Years, and the Rate of Interest, 4 per Cent. Then the Answer will appear to be .7599 multiply'd by 800, or 607.9200*l.* that is, 607*l.*: 18*s.*: 5*d.*, nearly.

PROBLEM II.

To find the present Value of an Annuity, certain, for a given Number of Years; according to any of the Rates of Interest specified in the preceding Problem.

SOLUTION.

Seek, in *Tab. IV.* the Number of Year's-Purchase answering to the given Term of Years; which, multiply'd by the proposed Annuity, gives the Answer.

EXAMPLE. Suppose the Annuity to be 100*l.* the Number of Years 7, and the Rate of Interest 4 per Cent. Then the Value sought will be 6.002 multiply'd by 100, or 600*l.*: 4*s.*

PROBLEM

P R O B L E M III.

To find (according to Observations on the Bills of Mortality of the City of London) the Probability, or Proportion of Chance, that a Person, of a given Age, continues in Being a given Number of Years.

SOLUTION.

Let the given Age be 40, and the Number of Years proposed 15.

Look, in *Tab. I.* against 40 Years, and also against 55 Years, the Age to which the Person must arrive, if he lives to the End of the given Term; corresponding to which you will find the Numbers 229 and 129, respectively; shewing that, of 229 Persons who attain to the Age of Forty, only 129 reach the Age of Fifty-five: Now the Excess of 229 above 129 being 100, it is evident that the Odds, or the Ratio of the Chances, for and against, surviving the proposed Term, will be as 129 to 100, or as 9 to 7, nearly: And, in the same manner, the Answer will be found in any other Case.

Note. Though This, and the following Problem, are not immediately concern'd about the Business of Annuities, yet they are the Foundation whereon the *whole* is grounded; and therefore do not improperly fill up the Place here allotted them.

P R O B L E M

PROBLEM IV.

To find (according to the foresaid Observations) the Number of Years of Life, which a Person, of a given Age, may, upon an Equality of Chance, expect to enjoy.

SOLUTION.

Seek the given Age, in Table II. and against it you will have the Answer, in Years and Decimal-Parts.

Thus it will appear that, a Person, 30 Years old, may, upon an Equality of Chance, expect 23.6 Years more, for his Share of Life *.

* By the Expectation, or Share, of Life, is not here to be understood, that particular Period, which a Person hath an equal Chance of surviving; this last being a different, and more simple Consideration. The Expectation of a Life (to put it in the most familiar Light) may be taken as the Number of Years at which the Purchase of an Annuity, granted thereon, without Discount of Money, ought to be valued. Which Number of Years will differ more or less from the Period above mention'd, according to the different Degrees of Mortality to which the several Stages of Life are incident.—Thus, it is much more than an equal Chance (according to the Table of the Probability of the Duration of Life, p. 254) that an Infant, just come into the World, arrives not to the Age of 10 Years; yet the Expectation, or Share of Life, due to It, upon an Average, is near 20 Years. The Reason of which wide Difference, is, the great Excess of the Probability of Mortality in the first, tender Years of Life, above That respecting the more mature, and stronger Ages.—If the Numbers that die at every Age were to be the same, the two Quantities above specified would also be equal; but when the said Numbers become continually less and less, the Expectation must, consequently, be the greater of the Two.

P R O B L E M V.

To find the Value of an Annuity for an assigned Life.

SOLUTION.

This Problem is resolved from Tab. V. by looking against the given Age, under the proposed Rate of Interest; corresponding to which you will have the Number of Years-Purchase required.

Ex. Let the given Age be Thirty-six, and the Rate of Interest 4 *per Cent.* and let the proposed Annuity be 250*l.* Then the Value thereof will appear to be 12.1 Years-Purchase, or 12.1 times 250*l.* Therefore, multiplying 250 by 12.1 (and cutting off one Figure, upon account of the Decimal) the Answer comes out 3025*l.*

After the same manner the Answer will be found in any other Case, falling within the Limits of the Table. But, as there may be Occasion, sometimes, to know the Values of Lives, computed at higher Rates of Interest than any There exhibited, the two following, practical, Rules are subjoined; by which the Problem is resolved, independent of Tables.

R U L E I.

If the given Age is not less than 45 Years (nor greater than 85) subtract it from 92; multiply the Remainder by the *Perpetuity**, and divide the Product by the said Remainder added to $2\frac{1}{2}$ times the *Perpetuity*; then the Quotient will be the Number of *Years Purchase* required.

* It may, perhaps, be of Use to some of my Readers, to be informed here, that, by the *Perpetuity* of an Estate, or Annuity, is understood the Number of Years-Purchase of the Fee Simple; found by dividing 100 by the Rate per Cent. at which Interest is reckoned.

EXAMPLE.

EXAMPLE.

Let the given Age be 50 Years, and the Rate of Interest 10 *per Cent.* Then, subtracting 50 from 92, there remains 42; which, multiply'd by 10, the *Perpetuity*, gives 420; this divided by 67, the Remainder increased by $2\frac{1}{2}$ times (10) the *Perpetuity*, quotes 6.3 nearly: Therefore, supposing the Annuity to be 100 *l.* its Value, in present Money, will be 630 *l.*

R U L E II.

If the given Age is less than 45 Years (but not less than 10) take $\frac{8}{10}$ of what it wants of 45; which divide by the Rate per Cent. increased by 1.2; then, if the Quotient be added to the Value of a Life of 45 Years, found by the preceding Rule, you will have the required Number of Years-Purchase in this Case.

EXAMPLE.

Let the proposed Age be 20 Years, and the Rate of Interest 5 *per Cent.* Here, taking 20 from 45, there remains 25; $\frac{8}{10}$ whereof is 20; which, divided by 6.2, quotes 3.2; and this added to 9.8, the Value of a Life of 45 (*found by Rule I.*) gives 13.0, for the Number of Years-Purchase that a Life of Twenty ought to be valued at.

It will be needless, I presume, to offer any thing farther by way of Example to the preceding Rules; which bring out the Conclusions so near the true Values, computed from *real* Observations, as seldom to differ therefrom by more than about $\frac{1}{10}$ or $\frac{2}{10}$ of an Year's-Purchase.

The Observations here understood are Those whereon the foregoing Tables are grounded (as the most proper Foundation for this *Place*) But, if any Person is desirous of seeing a similar Method of Solution accommodated to the *Breslau* Observations (publish'd by Dr. Halley, in N^o 196 of the Philosophical Transactions, which are considerably different from Those above-

276 *Of the Values of Annuities and Reversions,*

mentioned, deduced from the Bills of Mortality of the City of London) what follows may, perhaps, answer His Expectation. The Rule is thus, *Multiply the Difference between the given Age and 85 Years by the Perpetuity, and divide the Product by $\frac{8}{10}$ of the said Difference increased by twice the Perpetuity; then the Quotient will be the Answer.* Which, from the Age of Eight, to Eighty, will, for the general Part, come within less than $\frac{1}{8}$ of an Year's-Purchase of the Truth.

P R O B L E M VI.

To determine the Value of an Annuity, granted for a given Term of Years, upon the Contingency of its ceasing on the Failing of a proposed Life, if this should happen before the Expiration of the said given Term.

SOLUTION.

Find, in *Tab. VI. Column 1.* the Age of the assigned Life (or That nearest it), and find, in *Column 2.* the proposed Term of Years; against which last you will have the Answer.

EXAMPLE I.

Let the Age be 10, the Number of Years 15, and the Rate of Interest 5 per Cent. Then it appears, at one View, that the Value sought will be 9.4 Years Purchase.

EXAMPLE II.

Suppose the given Age to be 5 Years, the proposed Term 16 Years, and Interest as above: In which Case it appearing that the Value of a Term of 15 Years is 9.1, and That of a Term of 20 Years, 10.7, it is evident that the true Answer here must be about 9.5 — But, that there may be no Difficulty in allowing for the odd Years (which is the harder to do,
as

as the Differences are unequal) the Table at p. 264 is annexed, as a Supplement to that preceding it: To comprehend the Use of which the Difference of the Values, answering to the two nearest Tabular Numbers to the given Term (*in Tab. VI*) must be taken; which, in this Case, is 1.6: And then, by entering *Tab. VII. Column 1.* with this Difference, you will find against it, under the Excess (1) of the given Term above the next inferior Tabular Number, the Value 0.4 to be added to That answering to the said inferior Number in order to have the true Conclusion. — After the same manner the Value corresponding to the same Age, and a Term of 19 Years, will be found 10.4. But it may be proper to observe, that, to avoid Trouble, it will be sufficient, in most Cases, when the Age given cannot be exactly found in the Table, to take the Tabular Number that comes nearest to it, whether greater or lesser.

P R O B L E M VII.

To find the Value of an Annuity, for two assigned joint Lives, that is, for as long as they both continue in Being together.

SOLUTION.

Seek, in *Tab. VIII, Column 1.* the Age of the youngest Life (or That nearest to it), and find, in *Column 2,* the Age of the Elder; against which last you will have the Number of Years-Purchase required.

EXAMPLE I.

Let the two Ages be 20, and 35, Years, and the Rate of Interest 4 *per Cent.* Then it appears, at one View, that the Value sought will be 9.8.

278 *Of the Values of Annuities and Reversions,*

EXAMPLE II.

Let the proposed Ages be 25, and 37, Years, and Interest as before.

Here, if the Age of the Elder was to be $\left\{ \begin{smallmatrix} 35 \\ 40 \end{smallmatrix} \right\}$ the Value sought would be $\left\{ \begin{smallmatrix} 9.6 \\ 9.1 \end{smallmatrix} \right\}$

Therefore, when the Age is 37, it must be 9.4.

EXAMPLE III.

Suppose the given Ages to be 32 and 57, and the Interest of Money at 3 *per Cent.* In this Case, the Value corresponding to the Ages $\left\{ \begin{smallmatrix} 30 \\ 35 \end{smallmatrix} \right\}$ and $\left\{ \begin{smallmatrix} 57 \\ 57 \end{smallmatrix} \right\}$, appears to be $\left\{ \begin{smallmatrix} 7.6 \\ 7.4 \end{smallmatrix} \right\}$ Whence That answering to the given Ages 32 and 57, must be about 7.5.

But, in order to avoid Trouble, you may, upon Occasion, add an Year, or two, to one of the proposed Ages, and subtract as much from the Other, when they are nearly equal: But, if One of them much exceeds the Other, it will then be sufficient to take the nearest Number in the Table for the Lesser.

P R O B L E M VIII.

To find the Value of an Annuity, for the longest of two Lives, that is, for as long as either of them continues in Being.

SOLUTION.

Find, in Tab. IX, Column 1, the Age of the youngest Life (or That nearest it), and in Column 2, find

Note. *The Solutions to the several Problems inserted in the Course of this Work are, not improperly, divided into two different*

find the Age of the Elder; against which last you will have the Number of Years-Purchase required.

EXAMPLE I.

Let the given Ages be 15, and 40, Years; and let the Rate of Interest be 4 *per Cent.* Then it is apparent that the Value of an Annuity upon the longest of two such Lives will be 17.9, or nearly 18, Years-Purchase.

EXAMPLE II.

Let the Ages propounded be 25 and 62, and the Rate of Interest as before: In which Case we have 15.5 for the Value of the Annuity.

PROBLEM IX.

To find the Value of an Annuity for [three joint Lives, A, B, and C.

SOLUTION.

† Let A be the youngest, and C the oldest, of the three proposed Lives: Take the Value of the two *joint Lives* B and C (*by Tab. VIII.*) and find the Age of a single Life D, of the same Value (*by Tab. V.*) then find the Value of the *joint Lives* A and D; which will be the Answer.

*different Classes: The first whereof, hereafter distinguish'd thus *, are general, and strictly true, according to any Table of Observations, or Probability of Life what-so-ever: But the second Sort, marked thus †, are near Approximations only, yet such as are applicable, likewise, to any Table of Observation.*

EXAMPLE I.

Let the three given Ages be 20, 35 and 44 Years; and let the Rate of Interest be 3 *per Cent.* Then, the Value of the two oldest joint Lives B and C, will be found 9.0; answering to a single Life (D) of 61 Years: And the Value of the joint Lives A and D, which is 7.3 Years-Purchase, will be the Value sought.

EXAMPLE II.

Suppose the three Ages to be 17, 23 and 38, and the Rate of Interest 5 *per Cent.* Here the Value of the joint Lives B and C, will be 8.5; agreeing with That of a single Life (D) of 55 Years; whence the Value sought, or That of the joint Lives A and D, is in this Case 7.0, or, just 7 Years-Purchase.

PROBLEM X.

To find the Value of an Annuity for the longest of three Lives, A, B, and C.

SOLUTION.

† Let A be the youngest, and C the oldest, of the three proposed Lives. Find the Value of the joint Lives B and C (*by Tab. VIII*), and find (*by Tab. V*) the Age of a single Life, D, of the same Value: Moreover find (*by Tab. IX*) the Value of the longest of the Lives A and B, also That of the longest of the Lives A and C, and likewise That of the longest of the Lives A and D; then the last of these three Values, subtracted from the Sum of the two former, leaves the Value sought.

EXAMPLE I.

Let the three Ages be 20, 40 and 66 Years, and let the Rate of Interest be 4 *per Cent.* Then the Value

Value of the joint Lives B and C will be found 5.8; answering to a single Life (D) of 73 Years: Moreover the Value of the longest of the Lives A and B will be 17.2; that of the longest of the Lives A and C 16.0; and that of the longest of the Lives A and D 15.5. Therefore the Value sought is 17.7 Years Purchase.

EXAMPLE II.

Let the given Ages be 15, 28 and 37 Years, and the Rate of Interest as before: Here (proceeding as in *Prob. 7. Ex. 3.*) the Value of the two oldest joint Lives will be had 9.2, equal to That of a single Life (D) of $55\frac{1}{2}$ Years (*by Tab. V*). Hence, the three Values specified in the latter Part of the Rule will, in this Case, be 18.4, 18.0, and 17.4, respectively, and consequently the Value sought, just, 19 Years Purchase.

PROBLEM XI.

To find the Value of an Annuity granted upon three Lives, A, B, C, on Condition of its ceasing as soon as any two of them become extinct.

SOLUTION.

† Find, (*by Tab. VIII.*) the Value of each Pair of joint Lives, viz. of A and B, of A and C, and of B and C; then, from the Sum of those three Values, let twice the Value of the three joint Lives A, B, and C, (*found by Prob. IX*) be deducted, and the Residue will be the Answer.

EXAMPLE.

EXAMPLE.

Suppose the given Ages of A, B and C to be 20, 35 and 44 Years, respectively; and let the Rate of Interest be 3 *per Cent.*

Here the Value of the *joint Lives*

$\left\{ \begin{array}{l} \text{A and B} \\ \text{A and C} \\ \text{B and C} \end{array} \right\}$ will be $\left\{ \begin{array}{l} 10.9 \\ 9.6 \\ 9.0 \end{array} \right\}$. The Sum of which

three Numbers is 29.5; moreover the Value of the three *joint Lives*, A, B and C is, in this Case, 7.3. (See *Prob. 9. Ex. 1.*) Therefore 14.9 is the Value sought.

P R O B L E M XII.

The Value of an Annuity upon One, two, or three Lives, at the Rate of 4 per Cent. being known (from the preceding Problems); to find the Value of the same Life, or Lives, at the Rate of $3\frac{1}{4}$, $3\frac{1}{2}$, $3\frac{3}{4}$, $4\frac{1}{4}$, $4\frac{1}{2}$, or $4\frac{3}{4}$, per Cent.

SOLUTION.

† This Problem is solved by Means of Tab. X; from whence the Value that ought to be added to, or subtracted from, the given Value, at the Rate of 4 *per Cent.* is had by bare Inspection.

Thus it will appear that an Annuity, upon One, or more Lives, which at the Rate of 4 *per Cent.* is worth 16 Years Purchase, will, at the Rate of $3\frac{1}{2}$ *per Cent.* be worth 17 Years Purchase and three Tenths. This Method, though exact only, when apply'd to a single Life, is sufficiently near in the Case of two, or more, Lives.

REMARK.

REMARK.

As it is customary for an *Annuitant* to receive his Money half-yearly in equal Portions (instead of yearly) he hath, in this, a double Advantage; for, besides the Use of, each, first-half-yearly Payment for 6 Months, he also hath a Chance of receiving one Half-year's Value more, than if he was to be paid yearly. Now the Value of both these Considerations put together (let the Rate of Interest and the Number of Lives on which the Annuity depends be what they will) will always amount to $\frac{1}{4}$ of a Year's-Purchase. Therefore $\frac{1}{4}$ of a Year's Purchase must be always added to the Values found by the foregoing Problems, in order to have the true Answer, when the Payments are made Half-yearly. And, if the Payments are to be made Quarterly, then $\frac{3}{8}$ of a Year's Purchase ought to be added to the Value found by the preceding Calculations.

OF R E V E R S I O N S.

P R O B L E M XIII.

To find the Value of the Reversion of an assigned Life after a given Term of Years.

SOLUTION.

* From the Value of the proposed Life, subtract the Value of an Annuity for the given Term of Years, on the Contingency of its ceasing upon the Extinction of the foresaid Life ; the Remainder will be the Answer.

EXAMPLE.

A, aged 15 Years, expects to enter upon an Estate of 500 *l. per Annum*, after the Expiration of 10 Years ; which he is to hold thence-forward, during Life ; what is the present Value of his Expectation, reckoning Interest of Money at 4 *per Cent*.

Here the Answer will be 8.3 Years-Purchase, or 4150*l.* For the Value of the whole Life A being 15.8 (*by Tab. V*), and That of the first 10 Years 7.5 (*by Tab. VI*), the Difference will be 8.3, *as above*.

P R O B L E M XIV.

To find the Value of the Reversion of an Annuity, for the Remainder of a given Term of Years, after an assigned Life.

SOLUTION.

* From the Value of an Annuity certain for the given Term, subtract the Value of the Annuity for the said Term, on the Contingency of its ceasing upon the Failing of the proposed Life ; the Remainder will be the Value of the Reversion.

EXAMPLE.

EXAMPLE.

A, aged Twenty-five, who has the Right of an Annuity for 31 Years certain, makes over the Reversion thereof to B and his Heirs, to enjoy the same after his Decease, for the Remainder of the said Term. Now, in order to find the Value of B's Expectation, the Value of an Annuity *certain*, for 31 Years, is to be found; which, at the Rate of 4 *per Cent.* will be 17.58 (by *Tab. IV*). Moreover the Value of an Annuity for the same Term, on the Contingency of its Failing on the Extinction of A, will appear to be 12.9 (by *Tab. VI*). Therefore the Value sought, in this Case, is 4.68 Years Purchase.

P R O B L E M X V .

To find the Value of the Reversion of one Life after another.

SOLUTION.

* From the Value of the Life in Expectation subtract the Value of the two *joint Lives*; the Remainder will be the required Value of the Reversion.

EXAMPLE.

Let the Age of the Life in Possession be 55 Years, that of the Life in Expectation 20 Years, the Rate of Interest 5 *per Cent.* and the proposed Annuity 100 *l.* Then, by *Tab. VIII*, the Value of the two *joint Lives* will be 6.9; which, subtracted from 13.0, the Value of the Life in Expectation (*found by Tab. V*), leaves 6.1 Years-Purchase, for the Value of the Reversion. Which, multiply'd by the proposed Annuity, gives 610 *l.* its Value in present Money.

Note. In the Resolution of the preceding Problem, the Value that ought to be paid for putting in, or joining, a new Life to One already in Possession, is likewise determined

to add one Life to 1 note

determined. Thus, for Example, if A and his Heirs hold an Annuity of 100*l.* upon a single Life, of Fifty-five, and They would put in another Life, of Twenty, to hold the Annuity as long as either of the two Lives continue in Being; then the Sum which ought to be paid, as an Equivalent for that Consideration, is 610*l.* the very Value above exhibited.

P R O B L E M XVI.

To find the Value of the Reversion of two Lives after One.

SOLUTION.

* From the Value of the three Lives subtract the Value of the Life in Possession, the Remainder will be the Value of the Reversion.

EXAMPLE.

Let the Age of the Life in Possession be 40 Years, and the Ages of the two Lives in Expectation 20 and 66 Years; and let Interest be supposed at 4 *per Cent.* In which Case, the Value of the three Lives being 17.7 (*by Prob. 10, Ex. 1*), and That of the Life in Possession 11.5 (*by Tab. V*), the Answer comes out 6.2 Years-Purchase: So that, if the Annuity was to be 500*l.* the Value of the Reversion would be 3100*l.* Which Sum also expresses the Consideration that ought to be allow'd for putting in two Lives, of 20 and 66 Years, to a Life of Forty, already in Possession.

PROBLEM

P R O B L E M XVII.

To find the Value of the Reversion of one Life after two joint Lives.

SOLUTION.

* From the Value of the Life in Expectation, subtract the Value of the three joint Lives, there will remain the Value of the Reversion.

EXAMPLE.

Suppose the Age of the Life in Expectation to be 20 Years, and the Ages of the two Lives in Possession 35 and 44 Years; and suppose Interest of Money to be at 3 per Cent.

Here the Value of the three joint Lives will be found 7.3 (by Prob. 9, Ex. 1.) which being deducted from 17.2, there rests 9.9 Years-Purchase, for the Value of the Reversion.

P R O B L E M XVIII.

To find the Value of the Reversion of One Life after Two.

SOLUTION.

* From the Value of the three Lives, subtract the Value of the two Lives in Possession, there will remain the Value of the Reversion.

EXAMPLE.

Let 40 and 66 be the Ages of the two Lives in Possession, and 20 That of the Life in Expectation, and let the Rate of Interest be 4 per Cent.

Then (by Prob. 10, Ex. 1.) the Value of the three Lives will be found 17.7; from which taking, 13.0, the Value of the two Lives in Possession (found by Tab. IX.)

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to add one Life to two two

10 put a
Life in
a Lease

288 *Of the Values of Annuities and Reversions,*

the Remainder, 4.7, will be the Number of Years-Purchase at which the Value sought is to be estimated. Which, therefore, also expresses the Value that ought to be paid for putting in, or joining, a Life of 20 Years to two Others of 40 and 66 Years.

P R O B L E M XIX.

To find the Value of the Reversion of two joint Lives after One.

SOLUTION.

* From the Value of the two joint Lives in Expectation, subtract the Value of the three joint Lives, there will rest the Value of the Reversion.

EXAMPLE.

Let the Ages of the two Lives in Expectation be 20 and 44 Years, and That of the Life in Possession 35 Years; and let the Rate of Interest be 3 per Cent. Then, the Value of the three joint Lives being 7.3 (by Prob. 9, Ex. 1.) and That of the two joint Lives in Expectation 9.6 (by Tab. VIII) the Answer in this Case will be 2.3 Years-Purchase.

P R O B L E M XX.

Q and his Heirs, as soon as any two of three given Lives A, B, C, become extinct, are to enter upon an Annuity, in order to enjoy the same, during the Life of the Survivor: To determine the present Value of the Expectation.

SOLUTION.

* To the Sum of the Values of all the single Lives, add three times the Value of the three joint Lives; and from this Sum deduct twice the Sum of the Values of each

each Pair of *joint Lives* (*viz.* of A and B; A and C; and B and C) the Remainder will be the Answer.

EXAMPLE.

Let the three given Ages be 20, 35, and 44 Years, and the Rate of Interest 3 *per Cent.* Then the Values of the three *single Lives*, by *Tab. V*, will appear to be 17.2, 14.1, and 12.5. And, by *Prob. IX. Ex. 1*, the Value of the 3 *joint Lives* is found 7.3: The treble of which Value, added to the Sum of the three Former, gives 65.7. But the Values of the three different Pairs of *joint Lives* are found, by *Tab. VIII*, to be 10.9, 9.6, and 9.0: The Double of all which, taken from 65.7, gives 6.7 for the Number of Years-Purchase required.

P R O B L E M XXI.

A and B enjoy an Annuity, equally; betwixt them; which, after the Decease of either of them, is to belong intirely to the Survivor for Life; To find the Value of the Right of Each in that Annuity.

SOLUTION.

* From the Value of the Life A, or B, subtract Half the Value of the two joint Lives; the Remainder will be the Value of the Right of A, or B, accordingly.

EXAMPLE.

Let the Age of A be 25 Years, that of B 40 Years, and the Rate of Interest 5 *per Cent.* Then the Value of the two *joint Lives* will be 8.2 (*by Tab. VIII.*) whereof the Half is 4.1; this taken from 12.3, the Value of the Life A, leaves 8.2 for the Expectation of A; but, being taken from 10.3, the Remainder 6.2, will be the Value of B's Expectation.

P R O B L E M XXII.

A given Annuity, after the Decease of A, is to be divided equally between B and C, during their joint Lives; and then is to go intirely to the last Survivor for Life; 'tis propos'd to find the Value of B's Expectation.

SOLUTION.

* From the Value of the Reversion of the Life B after the Life A (*found by Prob. 15*) subtract half the Value of the Reversion of the *joint Lives* B and C after the Life A (*found by Prob. 19*), the Remainder will be the true Value of B's Expectation.

EXAMPLE.

Let the given Ages of A, B and C be 35, 20, and 44 Years, respectively; and let the Rate of Interest be 3 *per Cent.* Then the Value of the Life B after the Life A will appear to be 6.3; moreover the Value of the *joint Lives* B and C after the Life A will come out 2.3 (*Vid. Ex. to Prob. 19*). Therefore the Answer, in this Case, is 5.15 Years-Purchase.

P R O B L E M XXIII.

A, B, and C share an Annuity equally amongst them; which, upon the Decease of any One of the Three, is to be divided equally between the two Survivors during their joint Continuance, and then is to go intirely to the last Survivor for Life: To find the Value of the Right of A in the said Annuity.

SOLUTION.

* From the Value of the Life A, take half the Sum of the Values of the *joint Lives* A B, and the *joint Lives* A C; then to the Remainder add $\frac{1}{3}$ of the Value
of

of the three joint Lives A B C, and the Sum resulting will be the true Answer.

EXAMPLE.

Suppose the three Ages to be 17, 23, and 38 Years, respectively, and the Rate of Interest 5 per Cent.

Here the Value of the Life A will be 13.5; also that of the joint Lives A and B will be 10.1 (by Prob. 7) that of the joint Lives A and C, 8.6, and that of the three joint Lives A, B and C, 7.0 (by Prob. 9, Ex. 2). Therefore the Answer comes out 6.5 Years-Purchase.

Of successive Lives, and the Renewing of Leases.

P R O B L E M XXIV.

Supposing A to enjoy an Annuity for Life, and, at his Decease, to have the Nomination of a Successor, who is likewise to enjoy the Annuity for his Life; 'tis proposed to find the present Value of the two successive Lives.

SOLUTION.

* Multiply the Value of the Life A by the Value of the Life put in at his Decease, and divide the Product by the Perpetuity; then let the Quotient be subtracted from the Sum of the said Values, and the Remainder will be the Answer.

EXAMPLE.

Suppose the Value of the Life A, at 4 per Cent. to be 12 Years-Purchase, and that he hath the Liberty, at his Decease, to name a Life to succeed him of the greatest Value possible; which Value, according to Tab. V, is 16.4, answering to an Age of 10 Years. Therefore, in this Case, the Product of the two Values will be 196.8; which divided by 25, the Perpetuity, quotes 7.872; this, deducted from 28.4, the Sum of the said Values, leaves 20.528 Years-Purchase for the true Value of the two successive Lives.

P R O B L E M XXV.

To find the present Value of any Number of Lives in Succession.

SOLUTION.

* Multiply the Value of each Life (considered independent of the Rest, without Regard to Time, or Order of Succession) by the Rate of Interest, *per Cent.* dividing the Product by 100; subtract each Quotient from Unity, and multiply all the Remainders continually together, subtracting their Product, also, from Unity; then the last Remainder, multiply'd by the Perpetuity, will be the true Value of all the successive Lives.

Otherwise, thus.

* Find, in *Tab. IV.* the Number of Years answering to the Value of each of the given Lives; then find, in the same Table, the Value of an Annuity certain for the Whole of all those Years added together; which Value will be the Answer.

EXAMPLE.

Suppose the Number of Lives to be Three; whereof the Values, at 4 *per Cent.* are 8, 10 and 15 Years-Purchase. These being multiply'd, each, by 4, and the Products divided by 100, we have 0.32, 0.4, and 0.6, respectively: Which, subtracted severally, from Unity, leave 0.68, 0.6, and 0.4; whereof the continual Product is 0.1632; This being also taken from Unity, there rests 0.8368; which, multiply'd by 25, the Perpetuity, gives 20.92 for the true Answer.

Otherwise, by the second Method.

It appears that the Number of Years corresponding, in *Tab. IV.*

to $\left\{ \begin{array}{l} 8 \\ 10 \\ 15 \end{array} \right\}$ Years Purch. is about $\left\{ \begin{array}{l} 9.8 \\ 13.0 \\ 23.4 \end{array} \right\}$ whereof the Sum is 46.2;

against which we have 20.9, the same as before.

P R O B L E M

P R O B L E M XXVI.

A given Sum of Money, is to be received (as a Legacy), on the Decease of B, who is now of a given Age: What is the Value thereof in present Money?

SOLUTION.

* Subtract the Value of the Life B from the *Perpetuity*; then it will be, as the *Perpetuity* is to the Remainder, so is the proposed Sum, to its Value in present Money.

EXAMPLE.

Let the Age of B be 32 Years, the Rate of Interest 4 per Cent. and the given Sum 500 *l.* Then the Value of the Life B being 12.7, and the *Perpetuity* 25.0, it will be, as 25.0 : 12.3 :: 500 *l.* : 246 *l.* the Value sought.

P R O B L E M XXVII.

A given Sum is to be received on the Extinction of the first, second, or third, of three assigned Lives; to find the present Value of the Expectation.

SOLUTION.

* Find (by *Prob. 9, 10, or 11*) the Value of an Annuity granted upon the assigned Lives, on Condition of its ceasing on the Failing of the first, second, or third of those Lives, according to the Case proposed; which Value subtract from the *Perpetuity*, and then proceed as in the last Problem.

EXAMPLE.

Let the given Sum be 1000 *l.* the Rate of Interest 3 per Cent. and the Ages of the three Lives 20, 35 and 44 Years.

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Here

294 *Of the Values of Annuities and Reversions,*

Here the Value of an Annuity to cease upon Failing

of the $\left\{ \begin{matrix} 1^{\text{st}} \\ 2^{\text{d}} \\ 3^{\text{d}} \end{matrix} \right\}$ Life will be $\left\{ \begin{matrix} 7.3 \\ 14.9 \\ 21.3 \end{matrix} \right\}$ by Prob. $\left\{ \begin{matrix} 9 \\ 11 \\ 10 \end{matrix} \right\}$; which being

subtracted, severally, from the *Perpetuity* 33.3, there refts 26.0, 18.4, and 12.0; whence the Answer comes out 780 *l.* 552 *l.* or 360 *l.* according to the three fore-said Cases, respectively.

P R O B L E M XXVIII.

A purchases a Lease-hold Estate, upon One, two, or three assigned Lives, for a given Sum, on Condition that his Heirs fill up the Lease, continually, whenever all the Lives become vacant, paying a stated Fine: The Question is, to find the present Value of the whole Purchase allowed for the Estate, with the Annuity, or Rack-rent, answering thereto; supposing the Lives put in at each Renewal to be of the same, given, Ages.

SOLUTION.

* Find (by Prob. 5, 8, or 10.) the Value of the Life, or Lives, first proposed; find also (by the same) the Value of the Life, or Lives, with which the Lease is to be, constantly, renewed; subtract the former of These from the *Perpetuity*; then it will be as the Latter is to the Remainder, so is the Fine proposed to the present Value of all the Fines: Which added to the Sum, paid at entering, gives the Value allowed for the whole Purchase. This being multiply'd by the Rate of Interest, the Product, divided by 100, will be the Annuity, or Rack-rent corresponding.

EXAMPLE.

Let the Number of Lives be Three; whereof the Ages are 15, 28, and 37 Years; let the given Sum be 2000 *l.* the Fine 1000, and the Rate of Interest 4 per Cent. Then the Value of the three Lives will appear to be 19.0
(*Vid.*

(*Vid. Ex. 2, Prob. 10*), which, subtracted from 25.0, the Perpetuity, leaves 6.0: Therefore, if the Value of the Lives (found in like manner) with which the Lease is to be renewed, be taken at 17.7 Years-Purchase, we shall have $17.7 : 6 :: 1000\text{ l. } 339\text{ l.}$ Hence the whole Value allowed for the Purchase is 2339 l. and the Annuity corresponding appears to be 93.56 l. or 93 l. 11 s. 2 d.

PROBLEM XXIX.

Supposing A to purchase an Estate, in Copy-hold, upon any Number of assigned Lives, for a given Sum, on Condition that He and his Heirs fill up the Lease, continually, whenever a Life becomes vacant, by paying a proposed Fine: To find the present Value of the whole Purchase allowed for the Estate, with the Annuity, or Rack-rent, corresponding; supposing the Life put in, at each Renewal, to be of the same, given, Value.

SOLUTION.

* Subtract the Sum of the Values of all the, single, Lives, upon which the Lease is first granted, from the Perpetuity multiply'd by the Number of those Lives; then it will be, as the given Value of the Life to be put in at each Renewal, is to the Remainder, found above, so is the Fine proposed, to the present Value of all the Sums paid, for ever, for Renewing; which, added to the Value paid at first entering, gives the total Value of the Purchase. This being multiply'd by the Rate of Interest, the Product, divided by 100, will give the Annuity answering thereto.

EXAMPLE.

Let the Number of Lives be Three, and their Values (at 4 per Cent.) 10, 12, and 15 Years-Purchase; also let the Sum paid upon entering be 1600 l. and the proposed Fine 400 l. and suppose the Purchaser to have the Liberty of renewing with Lives of what Ages he thinks

296 *Of the Values of Annuities and Reversions,*

proper, or most to his own Advantage; which Ages, according to our *Table* at p. 260, appear to be between 7 and 12 Years; answering to 16.4 Years Purchase. Therefore, the Sum of the Values of the three first Lives being 37, and the Perpetuity 25, we have here $16.4 : 38 :: 400 : 926.8$, or 926 *l.* 16*s.* the present Value of all the Sums paid for renewing. Hence the whole Value allowed for the Purchase is 2526 *l.* 16*s.* and the corresponding Annuity 101 *l.* 1*s.* 5*d.*

P R O B L E M XXX.

The same being supposed as in the last Problem; to find how much the Rent-Roll of the first Proprietor's Estate ought to be increased, on account of the Fines paid at Renewing.

SOLUTION.

* Multiply the present Value of all the Sums paid for Renewing (*found as in the preceding Problem*) by the Rate *per Cent.* assigned; then the Product, divided by 100, will be the Answer. Which, in the Case, there proposed, will be found 37 *l.* 1*s.* 5*d.*

P R O B L E M XXXI.

There is an Estate, which, if A (who is a Minor) happens to die in a given Time, or before he attains to a certain given Age (suppose Twenty-one) is, after his Decease, to go to B and his Heirs for ever; to find the Value of B's Expectation.

SOLUTION.

* From the *Perpetuity*, subtract the Value of a Life of that Age to which the Expectation of B is limited; multiply the Remainder by the present Value of 1 *l.* to be received at the End of the given Term (*found by Tab. III*). And let this last Product be, again, multiply'd

ply'd by the Number of Persons, in *Tab. 1*, arriving to the Age above mentioned, dividing the Product thence arising by the Number, in the same Table, answering to A's present Age; then to the Quotient add the Value of the Life A, and subtract the Sum from the Perpetuity; the Remainder will be the true Value required.

EXAMPLE.

Let the Age of A be 8 Years, the proposed Estate 500*l. per Annum*, and Interest at 5 *per Cent*. Here the Value of a Life of Twenty-one (the Age to which B's Expectation is limited) being 12.9, and the *Perpetuity* 20, the Difference will be 7.1; which multiply'd by 0.5303, the present Value of 1*l.* to be received 13 Years hence (when A arrives to the Age of 21) gives 3.76513, or 3.76, nearly: This being multiply'd by 355, the Product, divided by 422, gives 3.16; to this adding 14.3, the Value of the Life A, and from the Sum deducting 20, the Perpetuity, there results 2.54, or $2\frac{1}{2}$ Years-Purchase, nearly, for the required Value of B's Expectation.

Of Reversions depending upon the Probability of one particular Life, in Possession, surviving the Rest.

PROBLEM XXXII.

B, who is of a given Age, will, if he lives 'till the Decease of A, whose Age is also given, become possessed of an Estate of a given Value; to find the Worth of his Expectation in present Money.

SOLUTION.

† Find (by *Tab. VIII.*) the Value of an Annuity on two equal joint Lives, whereof the common Age is equal to the Age of the older of the two proposed Lives and B; which Value subtract from the *Perpetuity*, and take Half the Remainder; then say, as the Expectation
of

298 *Of the Values of Annuities and Reversions,*

of the Duration of the younger of the two Lives A and B, found against the Age corresponding, in Tab. II, is to That of the Elder, so is the said Half-Remainder, to a fourth Proportional; which will be the Number of Years-Purchase required, when the Life B, in Expectation is the older of the Two. But, if B be the younger, then add the Value so found to That of the joint Lives A and B (found by Tab. VIII), and let the Sum be subtracted from the Perpetuity, and you will, also, have the Answer, in this Case.

EXAMPLE I.

Suppose the Age of A to be Thirty, that of B Forty, the Rate of Interest 4 per Cent. and the given Legacy 5000 *l.* or 200 *l.* per Annum. Then the Value of two equal joint Lives, of Forty, being 8.1, and the Perpetuity 25, the Remainder, or Difference, will here be 16.9; whereof the Half is 8.45: Therefore it will be, as 23.6 : 19.6 :: 8.45 : 7.02, Years-Purchase, or 1404 *l.* the required Value of B's Expectation.

EXAMPLE II.

Let the Age of A be Forty, and that of B Thirty, and the Rest as in the preceding Example. Here the Value of the joint Lives A and B will be 8.8; which added to 7.02 (found above), the Sum will be 15.82; whence the Answer, in this Case, is 9.18 Years-Purchase, or 1836 *l.*

P R O B L E M XXXIII.

C and his Heirs are intitled to an Estate of a given Value, upon the Decease of B, provided B survives A; to find the Value of Their Expectation in present Money.

SOLUTION.

† Find (by Tab. IX.) the Value of an Annuity upon the longest of two equal Lives, whereof the common Age

Age is That of the older of the Lives A and B ; which Value subtra $\text{\c{c}}$ t from the *Perpetuity*, and take Half the Remainder ; then it will be, as the Expectation of Duration of the younger of the Lives A and B (*found by Tab. II.*) is to That of the older, so is the said Half-remainder, to the Number of Years-Purchase required, when the Life B *is the older of the Two* : But, if B *be the younger* ; then, to the Number thus found, add the Value of an Annuity on the longest of the Lives A and B, and subtra $\text{\c{c}}$ t the Sum from the Perpetuity, for the Answer in this Case.

EXAMPLE.

Let the Age of A be Thirty, That of B Forty, the Rate propos'd 4 *per Cent.* and the given Estate 200 *l. per Annum.*

Here, the Value of the longest of two Lives, aged Forty each, will be found 15.0 ; which, taken from 25, leaves 10, for the Remainder. Therefore, it will be $23.6 : 19.6 :: 5 : 4.15$, the Number of Years-Purchase required ; answering to 830 *l.*

But, if A had been Forty, and B Thirty, the Value sought would have been 4.95×200 , or 990 *l.* Because the Value of the longest of the Lives A and B is 15.9 Years-Purchase.

P R O B L E M XXXIV.

C is to enter upon a given Annuity, for Life, on the Decease of B, in case B survives A : To determine the Value of his Expectation in present Money.

SOLUTION.

† Case 1^o. *If the Life C, in Expectation, be older than either of the other Two : Find, by Prob. 18, the Value of the Reversion of the Life C after the longest of the Lives A and B ; One Half of which will be the Answer.*

Case 2^o. *If the Life C be younger than one, or both, of the Other ; Then find the Reversion of the Life C after the*

300 *Of the Values of Annuities and Reversions,*

the longest of two equal Lives, of the same Age with the oldest of A or B; which multiply by the Expectation of Duration of the said oldest Life (*given in Tab. II*) and divide the Product by twice That of the Younger; so shall the Quotient be the Answer, *in Case B is older than A*: But, if B be younger than A, let the said Quotient be subtracted from the Value of the Reversion of the Life C after the longest of the Lives A and B, and the Remainder will be the Answer.

EXAMPLE I.

Let the Age of A be 30; That of B, 40; and That of C, 50 Years; and suppose the Rate of Interest to be 3 *per Cent.*

Then, *according to Case 1*, the Value of the longest of the three Lives will be found to be 19.6 (*by Prob. 10*). From which deducting 18.4 the Value of the longest of the two Lives A and B, the Remainder 1.2 will be the Value of the Reversion of the Life C after A and B; the Half of which, or $\frac{1}{10}$ of a Years-Purchase, is therefore the Value required.

EXAMPLE II.

Suppose the Ages of A, B, and C to be 66, 40, and 20, respectively, and the Rate of Interest 4 *per Cent.*

Here, *according to Case 2*, we are first to find the Value of the Reversion of a Life of Twenty after the longest of two equal Lives of 66; which (*by Prob. 18*) will come out 7.0: This being multiply'd by 10.1, and the Product divided by 39.2 (*vid. Tab. II*) there results 1.8 nearly. Which, as B is here younger than A, must be subtracted from (4.7) the Value of the Reversion of the Life C after the longest of the Lives A and B (*found by Prob. 18*) whence we get 2.9 for the true Answer in this Case.

PROBLEM

P R O B L E M XXXV.

C is to enter upon a given Annuity, for Life, on the Decease of B, provided the Latter is survived by A: To find the present Value of his Expectation.

SOLUTION.

† Find, by the last Problem, the Value of the Reversion of the Life C after the Life B, on the Contingency of B's surviving A; which, subtract from the absolute Value of the Reversion of the Life C after the Life B (*found by Prob. 15*); the Remainder will be the Answer.

EXAMPLE.

Let every Thing be supposed, as in the second Example of the last Problem: Then the former of the [above mentioned] Reversions will appear to be 2.9: And the Latter (by taking 92 the Value of the *joint Lives* B and C from 14.8, that of the Life C) will be found 5.6. Therefore 2.7 Years-Purchase is the Value sought.

P R O B L E M XXXVI.

A and B are possessed of an Annuity between them; which, if B survives A, is afterwards to be divided equally between B and C, during their joint Existence; and then is to go intirely to the last Survivor for Life: To find the Value of C's Expectation.

SOLUTION.

† Find, by *Prob. 34*, the Value of the Reversion of the Life C after the Life B, on the Contingency of B's surviving A; to which add Half the Value of the Reversion of the *joint Lives* B and C after the Life A (*found by Prob. 19*); the Sum will be the Value sought.

EXAMPLE.

EXAMPLE.

Suppose the given Ages of A, B, and C, to be 66, 40, and 20 Years, respectively, and the Rate of Interest 4 *per Cent*.

Then the Value of the former of the two Reversions above mentioned will be 2.9 Years-Purchase (*vid. Ex. 2, to Prob. 34*), and That of the Latter (by taking 5.2, the Value of the *joint Lives* A, B, and C from 9.2, the Value of the *joint Lives* B and C) will come out 4.0: Therefore 4.9 Years-Purchase is the required Value of C's Expectation.

P R O B L E M XXXVII.

C engages to pay to A a given Annuity, till such Time (if they both live so long) as the Latter comes to the Possession of an Estate, which he is intitled to on the Decease of B: In consideration whereof, A, on his Part, obliges himself, if he lives to enter upon the said Estate, to cause a given Annuity to be paid back to C for Life (but if either C, or A, happens to die before B, then the Money advanced by C is to be wholly lost to his Heirs). The Question is, to find the Advantage, or Disadvantage of either Party, in such a Contract.

SOLUTION.

† Find, by *Prob. 9*, the Number of Years-Purchase of the three joint Lives A, B and C; which multiply by the Annuity that A is to receive, the Product will be the Value of A's whole Expectation from C.

Find also, by *Prob. 34*, the Value of the Reversion of the Life C after the Life B, on the Contingency of B's surviving A; which subtract from the Value of the Reversion of the Life C after the Life B (*found by Prob. 15*); then the Remainder, multiply'd by the Annuity paid back to C, will give the Whole of C's Expectation from A: And the Difference of the two Values thus found will consequently be the Answer.

EXAMPLE.

EXAMPLE.

A young Gentleman (A) aged Twenty-five, having greatly disobliged his Father (B) a Gentleman of Sixty, to whom he is sole Heir, is forced to contract with a certain Person (C) aged 35, for an Annuity of 200*l.* (in order to a Support) which he engages to repay, after his Father's Decease, with another of 300*l.* to continue during the Life of C; *according to the Conditions specified in the Problem.*

Now, we are first to find the Value of the three joint Lives; which, at the Rate of 4 *per Cent.* comes out 5.6 Years-Purchase; and this, multiply'd by 200, gives 1120*l.* for the whole Value of A's Expectation.

Moreover, the Value of the Reversion of the Life C after the Life B, on the Contingency of B's surviving A, will come out one Year's Purchase, very near: And the Value of the Reversion of the Life C after the Life B (without Restriction) will appear to be 5.8 Years-Purchase (*by Prob. 7*). Hence 4.8, multiply'd by 300, which is 1440*l.* will be the total Value of C's Expectation: Who, therefore, gains 320*l.* by the Contract.

P R O B L E M XXXVIII.

C, if he lives till the Decease of B, is to receive a given Legacy, in case A is then extinct; to determine the Value of his Expectation in present Money.

SOLUTION.

† Case 1°. If the Life C be the oldest of the Three: From the Value of an Annuity on the Life C, take the Value of the two joint Lives B and C; multiply the Remainder by the given Sum, or Legacy, and divide the Product by twice A's Expectation of Duration (*found in Tab. II*); the Result will be the Value sought.

Case 2°. If the Life B be the oldest of the Three: Then, from the Value of an Annuity, for as many
Years

Years of C's Life as are expressed by the Double of B's Expectation of Duration (*found by Tab. II and VI*), subtract the Value of the two *joint Lives* B and C; multiply the Remainder by the given Sum, or Legacy, and divide the Product by twice the Expectation of A's Duration, *as in the preceding Case*.

Case 3°. *If the Life A be the oldest of the Three:* Then find the Value of the Life C, if older than B; otherwise, find (*by Tab. VI.*) the Value of as many Years Thereof, as are expressed by the Double of B's *Expectation of Duration*: And, from the Value thus found, let the Value of the *joint Lives* A and C be subtracted; multiply the Remainder by the given Legacy, then the Product, divided by twice B's *Expectation of Duration*, will be the Answer, in this Case.

EXAMPLE I.

Let the Age of C be 15 Years, that of B 70, and that of A 45; and let the proposed Legacy be 1000 *l.* and the Rate of Interest 3 *per Cent*.

This Example, it is plain, belongs to Case 2°. According to which we must first find B's Expectation of Duration (*by Tab. II.*) which appears to be 8.8, and the Double of it 17.6: Now (*by Tab. VI.*) the Value of 17.6 Years of C's Life is found 11.9; and (*by Tab. VIII.*) the Value of the joint Lives B and C appears to be 6.0; which last Value, subtracted from the Former, leaves 5.9; this multiply'd by 1000 (the given Legacy) produces 5900; which being divided by 35.6, the Double of A's Expectation of Duration, there results 165 *l.* for the required Value of C's Expectation.

EXAMPLE II.

Suppose the Age of C to be 15 Years, that of B 45, and that of A 70; suppose also the given Legacy to be 1000 *l.* and the Rate of Interest 3 *per Cent*.

Here, according to Case 3°, we must first seek, in *Tab. II.* for B's Expectation of Duration; this appears to

to be 17.8, and the Double thereof 35.6: Now the Value of 35.6 Years of the Life C (by Tab. VI.) will be found 16.7; from which subtracting (6.0) the Value of the Joint Lives A and C (found by Tab. VIII.) there remains 10.7; this, multiply'd by 1000, gives 10700; which, divided by 35.6, quotes 300 *l.* for the required Value of C's Expectation, in present Money.

P R O B L E M XXXIX.

Q, in case he lives till *R* and *S* are both extinct, is to receive a Legacy of a given Value; to determine the Worth of his Expectation in present Money.

SOLUTION.

† Find, by the last Problem, the Expectation of *Q*, on the Contingency of *R*'s surviving *S*; and also his Expectation, on the Contingency of *S*'s surviving *R*: The Sum of which Values will, consequently, be the whole of his Expectation, in the present Case.

EXAMPLE.

Let the Age of *Q* be 15 Years, that of *R* 45 Years, and that of *S* 70; also suppose the given Legacy to be 1000 *l.* and the Rate of Interest 3 *per Cent.*

Here the first Part of *Q*'s Expectation, depending on *R*'s surviving *S*, appears, from Ex. 2. of the preceding Problem (by substituting *Q* for *C*, *R* for *B*, and *S* for *A*) to be 300 *l.*

And the latter Part of his Expectation, depending on *S*'s surviving *R*, appears, from Ex. 1 (by substituting *Q* for *C*, *S* for *B*, and *R* for *A*) to be 165 *l.* Therefore the Answer is 465 *l.*

P R O B L E M XL.

Q and R, if they live till the Decease of S, are to receive, each of Them a given Legacy; but, in Case one of Them dies before Him, then the Whole is to go to the Survivor: To determine the Value of Q's Expectation.

SOLUTION.

† Find, by *Prob. 32*, the Value of Q's Expectation on the Sum assigned to himself (depending on the Chance of his surviving S, without further Restriction): Then find, by *Prob 38*, the remaining Part of his Expectation, on the Sum assigned to R (depending on the Contingency of S's surviving R:.) The Sum of the two Values so found will consequently be the Answer.

EXAMPLE.

Suppose Q to be 15 Years of Age, R 45, and S, 70; suppose also the Legacy of Q to be 600 *l.* That of R 1000 *l.* and the Rate of Interest 3 *per Cent.*

In the first Place we are to find the present Value of 600 *l.* (or an Annuity of 18 *l.*) to be received (or enter'd upon) by Q, at the Decease of S: Which Value comes out 23.37 Years-Purchase, or 420 *l.* Moreover, by *Ex. 1, to Prob. 38*, the latter Part of Q's Expectation, depending upon the Chance of his Receiving the 1000 *l.* assigned to B, will appear to be 165 *l.* So that the whole is 585 *l.*

PROBLEM

P R O B L E M XLI.

Q, R, and S share an Annuity equally among them; which, upon the Decease of any One of the Three, is to be divided equally between the two Survivors, during their joint Continuance; and then is to go intirely to the last Survivor and his Heirs, for ever: To determine the Value of the Right of Q in the said Annuity.

SOLUTION.

† First find, by *Prob. 39*, that Part of Q's Expectation which depends on his becoming possessed of the whole Annuity, on being the last Survivor: To which Value add half the Sum of the Values of the Joint Lives Q R, and Q S; and from the Aggregate let $\frac{2}{3}$ of the Value of the three Joint Lives Q R S be subtracted; the Remainder will be the Value sought.

EXAMPLE.

Suppose Q to be 15 Years of Age, R 45, and S 70; suppose moreover the Annuity to be 300*l.* and the Rate of Interest 3 *per Cent.* according to which the Value of the Perpetuity is 10000*l.* Whence, by the Example to the Problem above quoted, the former Part of Q's Expectation will appear to be 4650*l.* We are next to find the Values of the Joint Lives Q R, and Q S; These, by *Tab. VIII*, appear to be 9.6, and 6.0, Years-Purchase, respectively: Whereof the Half Sum is 7.8 Years-Purchase, or 2340*l.* Moreover, by *Prob. 9*, the Value of the three Joint Lives Q R S, will be found 5.2 Years-Purchase, or 1560*l.* and $\frac{2}{3}$ thereof 1040*l.* And so, by adding together 4650 and 2340, and taking 1040 from the Sum, the Answer comes out 5950*l.*

S C H O L I U M.

Wherein the Reasons of what is most material and difficult in the preceding Solutions, are explained.

The Solutions to the several Problems proposed in the foregoing Pages being deliver'd in a practical manner, without their Investigation (in order to render the Work as useful as possible, and acceptable to Those, to whom the Sight of an Algebraical Process would, at best, afford no satisfactory, or pleasing Idea) and there being, in the Number of the said Solutions, some few that, even, the Mathematical Reader (whom I should be loth to leave dissatisfy'd) may not readily see into the Reasons of; I shall here, according to my Promise, put down the Invention of what seems most material, and necessary to be explained. It is not, indeed, my Design to insist, in this Place, on the first Principles of the Laws of Chance, and the general Methods of Computation whereon the Subject is grounded; but, to *such* as want Information therein, shall take the Liberty to recommend my *Doctrine of Annuities*, printed in 1742. Nevertheless, as the Merit of a Work of this Nature, when apply'd to the common Uses of Life, consists in its Conformity with Truth and *real* Observations, it is but reasonable that, for the general Satisfaction of Those into whose Hands this Tract may happen to fall, something should be, first of all, said relative to the Observations whereon the foregoing Tables have their Foundation.

The first of these Tables (shewing the Probability of the Duration of Life) on which the Rest depend, was computed from 10 Years Observations on the Bills of Mortality of the City of *London*; and is the very same, in effect, with that printed in my Book of Annuities above mentioned; tho' adapted to a different Radix. Both these Tables agree with That first of all published by Mr. *Smart*, whom I have followed, except in very low Ages; where it seem'd necessary to make some Alterations.

Alterations. I am sensible, indeed, that these Alterations have been condemned, as arbitrary and without Foundation: But that some such Allowance ought to be made, on account of the great, and unequal, Afflux of People, of different Ages, to Town, is, I think, sufficiently evident, from the following Reasons.

First, it is to be observed, agreeable to what I have elsewhere advanced, that the said continual Resort of Strangers to Town, would no way influence the Values of Annuities deduced from Observations on the Bills of Mortality, if Those arriving and settling *There*, at the several different Ages of Life, were to be in the same Proportion as the whole Number of the Living of the same Ages; since it is, not on the Greatness of the Numbers that die at each particular Age, but on their Ratio that the whole of the Business depends.

To render this as obvious as may be, by a familiar Example; suppose A and B to be two equal Cities, with regard to the Number of their Inhabitants, and let Half the Inhabitants of the Former, who are above the Age of Twenty, be supposed to remove to the Latter; then it is plain that the Bills of Mortality of B, for all Ages above Twenty, will be immediately increased by One Half; and so the same Proportion still preserved: For the Equimultiples of Quantities are to one another as the Quantities themselves.

But, with respect to Ages below Twenty, the Bills still continuing the same, the Numbers here cannot be compared with Those above, in order to obtain the true Ratio of the Probabilities of Mortality at the different Ages of Life; seeing These last are too little by One Half.

It is evident from hence, that, if there be any Part of Life wherein the Number of Those that remove to Town falls short of the Proportion above specified, the Bills of Mortality, for that Interval, will not truly exhibit the Probability of Mortality without some Allowance or Correction.

Now it is certain, from manifold Experience, that very few Persons come to live in Town under the Age

310 *Of the Values of Annuities and Reversions,*
of Fifteen, in comparison of the Numbers that arrive
There, between the Ages of Fifteen and Thirty;
though the Number of all the Living comprehended
in the former Period is much greater than That in the
Latter.

Add to this, that the Bills of Mortality, with Re-
spect to small Ages, are also lower than they would
otherwise be, on account of a great Number of Youth,
of the Better Sort, who are sent into the Country
for the Benefit of Air and Education. These, at
their Return, together with the Arrival of a Multitude
of Working People (who, after having served an Ap-
prenticeship in the Country, are willing to learn Ex-
perience, and try their Fortune in Town) very much
increase the Body of Inhabitants: And it is chiefly
to this Consideration that the great Increase in the
Bills of Mortality after the Age of Twenty, like a
Rivulet swoln by a sudden Rain, is to be ascribed: For I
dare affirm, that there is Nothing in Man's Constitution
whereby so great a Disproportion during so short an In-
terval, can be, physically, accounted for: Nor does it
seem, in the least, Reasonable to suppose that a Person,
of twenty-five Years of Age, has more than 4 times
a greater Chance to die within the Compass of an Year,
than One of fifteen; and yet this is the Case according
to the Table, I have been condemned for altering—
As to the Reason of my Beginning the Table with the
Number 1280 (instead of 1000), it was with no other
Intent than to avoid Trouble in making, what I judged,
the necessary Alterations. Nor does it at all signify what
Number is made the Radix of a Table of this Kind, pro-
vided the same Proportions are preserved throughout.
However, in conformity to Custom, I have here been
at the Pains to reduce the said Table to the com-
mon Radix.

Amongst Those who have been severe on the above
Alterations, there is a certain Foreign Gentleman,
M. Parcieux, Member of the Royal Academy of
Sciences at *Paris*, whom I must not pass by without
a more particular Notice. This Gentleman, in a Book
intitled

intituled *Essai sur les Probabilités de la vie humaine*, (as appears by an Account given of it in the *Memoirs* of the said Academy, 1746) accuses me of *Having assigned much too great a Probability of Mortality to the youngest Ages*; and further affirms that *this Error* (for such He calls it) *arises from omitting to include, in the Account, those Persons who, having escaped the Weakness of Infancy, go out of the Bills of Mortality, and die elsewhere.*

Whether, or no, the Fact be as it is here represented, this Writer does not appear to be a sufficient Judge: He does not seem to be appriz'd, that there is not, perhaps, a City in the World so fatal to the Infant State as the City of *London* (owing, too much, to the Intemperance of Parents and profuse, irregular manner of Living). Besides, were the Bills of Mortality of a Place to be ever so regular, and well kept, it ought to be remembered that there can be no Arriving at an absolute Certainty, in Matters of this Nature. Now, were it for these Reasons alone, I could not help looking upon this Gentleman's Assertions (to view Them in the most favourable Light) as a little too peremptory; especially where He tells the World that, *I was not aware, that, by avoiding Sir W. Petty's Fault, I my self fell into the contrary Mistake.* I declare, I am not conscious of any such Mistake; nay I can positively affirm, that the Reason he assigns for it, is altogether chimerical: Nor is there any thing I have said in my Preface, or elsewhere, that can justify such a Construction. But, what need have I to insist upon this! Every body Here knows that the Number of Those, who are born in Town, and die elsewhere, is inconsiderable in comparison of the Number of Those who, on the contrary, leave the Country and die in Town: So that, had they been, *actually*, disregarded, no such Error as He speaks of could possibly have arisen from thence.

The second Table, in this Work, *shewing the Number of Years that a Person of any Age, may, upon an Equality of Chance, expect to enjoy*, is deduced from the First; by a Method of Computation no way different

312 *Of the Values of Annuities and Reversions,*

from That, for finding the Value of an Annuity upon a single Life, when Money is supposed to bear no Interest; only the Value of one Half-Year, extraordinary, is to be added *here*; because an *Annuitant*, who receives his Money yearly, has an equal Chance of living Half a Year after the Time of the last Payment.

As to the Manner of computing the Tables, for the Values of single, and joint, Lives; this I have shewn (with proper Expedients to facilitate the Trouble of such a Work) in my Book of Annuities aforesaid, to which I must beg Leave to refer. — The first eight Problems, relating only to the Use of these Tables, need no other Explanation than is already given them.

The 9th Problem, *determining the Value of three joint Lives*, is the first that seems necessary to be taken Notice of *here*: Whereof the Solution is not, indeed, strictly conformable to the Table of Observations, being only an Approximation, but such an One as answers very near the Truth; and which may be apply'd with Advantage to, almost, any Hypothesis, or Table of Observations. The Reasonableness of the Method of proceeding is evident from the Nature of the Subject, without calling in the Assistance of any Kind of Computation. And, in a Number of Examples, respecting Lives of different Ages, I scarce ever found the Error to exceed $\frac{1}{8}$ of an Years-Purchase.

The Solution to our 10th Problem, *for determining the Value of the longest of three Lives*, is equally exact, and built upon the same Foundation, with the preceding One. For, the Value of the longest of three Lives being truly, and universally, expressed by $A + B + C - (AB) - (AC) - (BC) + (ABC)$; (where (AB) , (AC) , and (BC) denote the Values of each Pair of *joint Lives*, &c. *Vid. Doct. Ann. p. 23*); it follows that the said Value, if (according to *Prob. 9*) a single Life D be taken equal in Value to the two *joint Lives* (BC) , will also be expressed by $A + B + C - (AB) - (AC) - D + (AD)$
or

or by its Equal $A + B - (AB) + A + C - (AC) - A + D - (AD)$: But $A + B - (AB)$ is the Value of the longest of the two Lives A and B, and $A + C - (AC)$, that of the longest of the Lives A and C, &c. Whence the Whole is manifest.

The Solution to the next Problem, and Those that follow afterwards on Reversions, and successive Lives, are either such as are evident from plain Reasoning, exclusive of Mathematical Principles, or such, whose Demonstrations I have already given elsewhere: For which Reasons it seems needless to say any Thing further about them in this Place.

But, in those Kinds of Reversions consider'd in the Thirty-second, and the following Problems, depending upon the Probability of one particular Life, in Possession, surviving the Rest, the Reader must, I apprehend, have met with some Difficulty, the Rules there laid down being derived by Virtue of a certain Hypothesis; which, however, answers very near the Truth.

In this Hypothesis the Numbers of the Living at each several Age, differing by an, equal, Interval of one Year, are supposed to form an Arithmetical Progression (or a Series whose Terms decrease continually by a common Difference). This Assumption was first adopted by Mr. *De Moivre*, and apply'd by him, to good Purpose, in calculating the Values of Lives, according to the *Breslaw* Observations (*mentioned in p. 275*), with which it, indeed, much better agrees than with Those deduced from the Bills of Mortality of the City of *London*.

But, though the Hypothesis aforesaid cannot be apply'd with equal Advantage Here, without some farther Consideration, seeing the Differences of the Numbers in *Tab. I*, are far from being the same throughout; yet, by taking the Mean of the said Differences, the Value of a *real* Life may be converted into That of an equal *imaginary* One, conformable to the said Hypothesis. This

314 *Of the Values of Annuities and Reversions,*

This is actually done, in *Tab. II.* in which you have the Number of Years that a Person of any Age, may upon an Equality of Chance, be expected to enjoy: The Double whereof is the *Complement* of an equal, *imaginary* Life, according to the said Hypothesis; where, by the *Complement*, is to be understood the Number of Years that *such* a Life has a Chance of continuing in Being.

Thus it will appear, that the Value of a *real* Life, of Fifty, is equal to That of an *imaginary* One, whose *Complement* is 32 Years. This indeed is exact, only, when Money is supposed to bear no Interest; in all other Cases, the Latter is only an Approximation, but so near as to come within a sufficient Degree of Exactness, as will be seen from the annexed Table, wherein are exhibited the Values of Annuities for every tenth Year of Life, both according to real Observation, and to the said Hypothesis.

Age.	Value at 5 per cent. by Obs.	Value at 5 per cent. by Hyp.	Value at 4 per cent. by Obs.	Value at 4 per cent. by Hyp.	Value at 3 per cent. by Obs.	Value at 3 per cent. by Hyp.
10	14.3	14.2	16.4	16.3	19.0	19.1
20	13.0	13.1	14.8	15.0	17.2	17.2
30	11.6	11.9	13.1	13.4	15.0	15.1
40	10.3	10.7	11.5	11.9	13.2	13.4
50	9.2	9.6	10.1	10.4	11.4	11.5
60	7.9	8.1	8.4	8.6	9.2	9.4
70	6.2	6.3	6.5	6.6	6.9	7.1

By inspecting of which Table it will appear, that the Error seldom exceeds $\frac{1}{10}$ Part of the whole Value, and is the least of all, where the Interest of Money is lowest, agreeable to what was before intimated.

But before I proceed to the Use of the foregoing Hypothesis in the Resolution of the several Problems
above

above mentioned, I shall put down the Investigation of a known Theorem, for finding the Value of any proposed Life, according to the said Hypothesis: Which (being done without the Summation of Series's) may perhaps appear more obvious than That delivered, at p. 15, of my former Tract.

Let, therefore, a be taken to denote the Complement of the proposed Life A (as above defined), also let P denote the Value of the Perpetuity, and r the Amount of 1% in one Year: And, in order to render the Operation as easy as may be, suppose another Person (Q) and his Heirs to be intitled to the Reversion of the Annuity, for ever, after the Demise of A : Then, if the Value of the said Reversion be computed, and subtracted from the Perpetuity, the Remainder will, evidently, be the required Value of the Life A . But, to find the true Value of this Reversion, we must determine the several Parts Thereof, which depend on the Contingency of A 's becoming extinct in the first, second, third, and each succeeding Year.

Thus, if the Life A fails the first Year, whereof the Probability (according to the Hypothesis) is $\frac{1}{a}$, then, as Q and his Heirs become possessed of the whole Estate, or Perpetuity (without any Deduction) it is plain that the true Value of their Expectation, on the Contingency of A 's becoming extinct the first Year, will be $\frac{1}{a} \times P$.

Again, if the Life A fails the second Year, whereof the Probability is also $\frac{1}{a}$, the Value of Q 's Expectation will be the Value of the Perpetuity, discounted for one Year: And so the Expectation on this Event is $\frac{1}{a} \times \frac{P}{r}$.

And, in the same manner, it will appear, that the Expectation, on the Contingency of A 's dying in the 3^d, 4th, 5th, &c. Year, will be represented by $\frac{1}{a} \times$

316 *Of the Values of Annuities and Reversions,*

$\frac{P}{r^2}$, $\frac{1}{a} \times \frac{P}{r^3}$, $\frac{1}{a} \times \frac{P}{r^4}$, &c. respectively; and consequently, that the Sum of all These, or its Equal $\frac{rP}{a}$ into $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} + \dots + \frac{1}{r^a}$, will be the

total Value of the Reversion. But the Series $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \dots + \frac{1}{r^a}$ is known to express the Value of an Annuity, *certain*, for a Years: Which let be denoted by M ; then, the Value of the Reversion being $= \frac{rP}{a} \times M = \frac{P + 1 \times M}{a}$ (because $P = \frac{1}{r-1}$) the Value of the proposed Life A, in Possession, will consequently be $P - \frac{P + 1 \times M}{a}$. Q. E. I.

I come now to the Investigation of the Solutions of the Thirty-second, and the succeeding Problems.

In order to which let a and b be taken to represent the Complements of two imaginary Lives, of the same Values with the two real Ones A and B (*Vid. Prob. 32*). Moreover, let S denote the given Sum, or Legacy, to be received on the Decease of A, if B be then living; and let r represent the proposed Rate, or 1% increased by its Interest for one Year.

Now, to obtain an Answer to the said Problem, we must find the Value of B's Expectation on each particular Year, according to the Probability he has of receiving the Legacy (S) at the End of that Year: And each of these Values must be, again, consider'd in two Parts, according to the Chance of both the Lives Failing, or not Failing, within the Compass of the same Year.

The Probability that both A and B are extinct at the End of the first Year being $\frac{1}{a} \times \frac{1}{b}$, the Expectation

thereon

thereon would be $\frac{1}{a} \times \frac{1}{b} \times \frac{S}{r}$ (or $\frac{S}{abr}$) was it a

Certainty that A would drop first; but as there is an equal Chance for the Contrary (under the above Restrictions), the true Expectation can therefore be only

$\frac{S}{2abr}$. In the same manner, the Expectation, on the

Chance of both the Lives failing in the 2^d, 3^d, 4th, &c. Years, will be $\frac{S}{2abr^2}$, $\frac{S}{2abr^3}$, $\frac{S}{2abr^4}$ &c. And the

Sum of all These $\frac{S}{2ab} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} + \text{&c.}$

will consequently be the whole of B's Expectation, on the Contingency of both the Lives dropping in one, and the same, Year: Where the Series is to be continued to as many Terms as are expressed by the Complement *b* of the oldest Life.

But now, with regard to the other Part of B's Expectation, depending on the Probability of his surviving the Year wherein A expires (which is, by far, the most considerable), it is evident that the Part thereof which depends upon the Chance of his coming into Possession at the End of any proposed Year, is compounded of the Probability of his being then alive, and That of A's dying within the Compass of the same Year (allowing also for the Discompt of Money to that Time).

Hence the Expectation on the proposed Sum *S*, on the Contingency of receiving it at the End of 1, 2, 3,

4, &c. Years, will be expressed by $\frac{b-1}{b} \times \frac{1}{a} \times \frac{S}{r}$,

$\frac{b-2}{b} \times \frac{1}{a} \times \frac{S}{r^2}$, $\frac{b-3}{b} \times \frac{1}{a} \times \frac{S}{r^3}$, &c. respectively.

The Aggregate of all which Values, added to the former Part of B's Expectation (found above) gives

$\frac{S}{a} \times \frac{2b-1}{2br} + \frac{2b-3}{2br^2} + \frac{2b-5}{2br^3} + \frac{2b-7}{2br^4}$ &c. for the

whole

318 *Of the Values of Annuities and Reversions,*

whole Value required. Where the Series is to be continued to as many Terms as there are Units in b , the Complement of the oldest Life.

Now, seeing the Conclusion, here brought out, is not otherwise affected by a than its being a general Divisor to the *whole*, it is evident that the true Value of the Expectation, supposing b to remain the same, will be reciprocally as the Complement (a) of the younger Life, and therefore, also, as the Expectation of its Duration, which is the Half thereof. But the said Value, when $a = b$, is in the same Proportion to the given Sum S , as Half the Reversion of an Annuity, for ever, after two such equal *joint Lives*, is to the *Perpetuity* (*Vid. Doct. Ann. p. 76.*) Whence the Reason of the Solution to *Prob. 32* is manifest. And, by proceeding according to the very same Method, the 33^d will also appear conspicuous.

But, as to the 34th, the Exegesis will be something more complex; since we must, first of all, determine *the Probability that both A and B will be extinct at the End of any Number of Years, on the Contingency of A's dying first.*

In order to which, let a and b be taken as above specified; and suppose that a Sum of Money S is to be received on the Decease of B (whenever that happens) provided A is then extinct.

Now (as before) the Probability of receiving the Sum S , within the Compass of any proposed Year, must be considered in two Parts, according to the Chance of One, or both Lives, failing in that Year.

Thus the Probability of receiving it the 2^d, 3^d, or 4th, &c. Year, on the Contingency of A's being dead before the Commencement of that Year, will be defined by $\frac{1}{a} \times \frac{1}{b}$, $\frac{2}{a} \times \frac{1}{b}$, or $\frac{3}{a} \times \frac{1}{b}$, &c. respectively.

And the Probability of receiving it, any Year, on the Contingency of both the Lives failing within that Year, will, constantly, be expressed by $\frac{1}{2ab}$,
for

for as long as They both have a Chance of continuing together.

Hence the whole Probability, depending on the 1st, 2^d, 3^d, 4th, &c. Year (by adding $\frac{1}{2ab}$ to each of the

Values first found) appears to be $\frac{1}{2ab}$, $\frac{3}{2ab}$, $\frac{5}{2ab}$,

$\frac{7}{2ab}$, &c. respectively : Therefore the Sum of all These,

continued to as many Terms (n) as there are Years proposed, will, evidently, be the true Value of the Probability required. But the Sum of n Terms of the Series, 1, 3, 5, 7, &c. is $= n^2$: Therefore the required Probability is also truly expressed by $\frac{n^2}{2ab}$.

But here we must not omit to take Notice, that this Conclusion holds, only, when (n) the Number of Years is less than the Time that A and B have a Chance of living together : For, if n be greater than a , and the Age of A, at the same time, exceeds That of B ; then, besides the Probability of Receiving the Sum S , in the Time (a) during which A hath a Chance of living, there is a further Chance, or Probability, depending on the Contingency of receiving it, or of B's dying, after the Expiration of the said Time : Which (as the Life A is then out of the Question) will consequently be expressed by $\frac{n-a}{b}$. And this added to $\frac{aa}{2ab}$,

found above (by writing a for n), gives $\frac{2n-a}{2b}$, for the true Measure of the required Probability in this Case, where n is greater than a .

Now, to apply these Conclusions to the Problem in Question, let c denote the Complement of the Life C : Then, as the Probability of his receiving the Rent of any proposed Year, is compounded of the Probability of his living till then, and of the Probability that both A and B are extinct, with the Restriction of A's dying first (as above determined) ; it is manifest that the present Value

320 *Of the Values of Annuities and Reversions,*

Value of his Expectation depending on the 1; 2, 3, 4th, &c. Years Income, will be expressed by $\frac{1^2 \times c - 1}{2abc}$ &

$\frac{2^2 \times c - 2}{2abcr^2}$, $\frac{3^2 \times c - 3}{2abcr^3}$, $\frac{4^2 \times c - 4}{2abcr^4}$, &c. respectively.

And so the whole Expectation, or the Value sought, will be truly exhibited by the following Series $\frac{1}{2abc} \times$

$$\frac{1^2 \times c - 1}{r} + \frac{2^2 \times c - 2}{r^2} + \frac{3^2 \times c - 3}{r^3} + \dots + \frac{c^2 \times c - c}{r^c},$$

when C (according to Case 1 of the Solution) is older

than either A or B; and by $\frac{1}{2abc} \times \frac{1^2 \times c - 1}{r} +$

$$\frac{2^2 \times c - 2}{r^2} + \dots + \frac{b^2 \times c - b}{r^b} + \frac{b^2 \times c - b - 1}{r^{b+1}} + \dots + \frac{b^2 \times c - c}{r^c},$$

when C is younger than B, and older than A (according to the former Part of Case 2).

But the first of these two Values is known to express Half the Value of the Reversion of the Life C after the longest of the Lives A and B.

And, with regard to the Second, it is evident (because

$$\frac{1}{bbc} \times \frac{1^2 \times c - 1}{r} + \frac{2^2 \times c - 2}{r^2} + \dots + \frac{b^2 \times c - b}{r^b} + \frac{b^2 \times c - b - 1}{r^{b+1}} + \dots + \frac{b^2 \times c - c}{r^c}$$

is the Value of the Reversion of the Life C after the longest of two equal Lives B, B) that the true Value Thereof must be equal to That of the said Reversion, multiply'd by $\frac{b}{2a}$: Hence the former

Part of Case 2^o is also manifest. — As to the latter Part, it is derived from thence, by plain Reasoning. For, since the absolute Value of the Reversion of the Life C after the two given Lives A and B, may be considered as composed of two Parts, depending on the
Chance

Chance of the Older of the two Lives surviving the Younger, and on the Chance of the Younger's surviving the Older; it is very easy to perceive that, if the former of these Parts be taken from the Whole, the other Part will remain.

Thus much concerning the Solution of our 34th Problem; on which the three succeeding Ones, almost, intirely depend; and therefore it will be unnecessary to dwell upon them here.

The Solution to the 38th may, however, need some sort of Explanation: In regard to which it is evident, that the Probability of the Expectant's receiving the proposed Legacy (S) at the End of any particular Year, is compounded of the Probability of his continuing alive till then, and That of B's dying within the Compass of the said Year, on the Contingency of A's being extinct before. Now the last of These is already

found to be $\frac{1}{2ab}$, $\frac{3}{2ab}$, $\frac{5}{2ab}$, &c. according as the 1st, 2^d, 3^d, &c. Year is assigned; whence it is plain that the Expectation, on the Contingency of coming into Possession at the End of the first, second, third, &c. Year, will be $\frac{1 \times c - 1 \times S}{2abcr}$, $\frac{3 \times c - 2 \times S}{2abcr^2}$, $\frac{5 \times c - 3 \times S}{2abcr^3}$, &c. respectively;

and, consequently, that the Sum of all These, or $\frac{S}{2bac} \times$

$$\frac{1 \times c - 1}{r} + \frac{3 \times c - 2}{r^2} + \frac{5 \times c - 3}{r^3} + \frac{7 \times c - 4}{r^4} + \text{\&c.}$$

will be the, whole, Value required.

But this Series may be divided into two Others,

viz. $\frac{S}{abc} \times \frac{1 \times c - 1}{r} + \frac{2 \times c - 2}{r^2} + \frac{3 \times c - 3}{r^3} + \frac{4 \times c - 4}{r^4} + \text{\&c.}$

and $\frac{-S}{2abc} \times \frac{c-1}{r} + \frac{c-2}{r^2} + \frac{c-3}{r^3} + \frac{c-4}{r^4} + \frac{c-5}{r^5} + \text{\&c.}$

Y

And

322 *Of the Values of Annuities and Reversions,*

And the former of These may be, again, resolved

$$\text{into } \frac{S}{a} \times \frac{c-1}{cr} + \frac{c-2}{cr^2} + \frac{c-3}{cr^3} + \frac{c-4}{cr^4} + \frac{c-5}{cr^5} \&c.$$

$$\text{and } \frac{-S}{a} \times \frac{b-1 \times c-1}{bcr} + \frac{b-2 \times c-2}{bcr^2} + \frac{b-3 \times c-3}{bcr^3} \&c.$$

whereof the last (exclusive of the General Multiplier) is known to express the Value of the joint Lives B and C: And the preceding Part (or Series) is likewise known to represent, either the Value of the *whole* Life C, or else the Value of as many Years of it as are expressed by the Complement of the oldest of the Lives A and B, according to the different Cases specified in the Solution of the Problem: Whence the

Reason of the Whole is apparent.—As to the Series $\frac{-S}{2abc}$

$$\times \frac{c-1}{r} + \frac{c-2}{r^2} + \frac{c-3}{r^3} \&c. \text{ whereof no Notice is taken}$$

in the Solution, it is so small, in Comparison of the other Two, that to have embarrass'd the Solution with it would not only have been unnecessary, but quite improper, considering the Nature of the Subject; where an absolute Exactness (for want of knowing the precise Law of the Decrements of Life) is impossible.

There remain yet three other Solutions to speak of; but as These depend intirely on the 38th, and some of the preceding Ones, it will be needless to say any thing further about Them in this Place. I have, already, been very particular on these Kinds of Problems, and the more so, as there is no Method yet published (that I know of) by which They can be rightly determined. 'Tis true the Manner of proceeding by first finding the Probability of Survivorship (which Method is used in my former Work, and which a celebrated Author on this Subject has largely insisted on, in three successive Editions) may be applied to good Advantage, when the given Ages are nearly equal: But then

then, it is certain, that this is not a genuine Way of going to Work ; and that the Conclusions hence derived are, at the best, but near Approximations. The Rate of Interest that Money bears must be compounded with the Probability of Survivorship, and the Expectation on each particular Year must be determined, in order to have a true Solution.

I shall conclude what I have to say on this Subject of Annuities with the Solutions of two, or three, of the most useful Problems, according to a Method very different from That whereby They are usually investigated.

P R O B L E M I.

To determine the present Value of an Estate in Land, for a single Life A, of a given Age, according to the Hypothesis of an uniform decrease of the Probability of Life.

Let r denote the Rate, p the Perpetuity, and a the Complement of the proposed Life. Moreover let x (considered as flowing uniformly) be any Time elapsed from his entering into Possession: Then the present Value of the Expectation on the next succeeding

Moment \dot{x} , will, evidently, be expressed by $\frac{a-x}{a} \times$

$\frac{\dot{x}}{r^x}$, or, its Equal, $\dot{x} q^x - \frac{x \dot{x} q^x}{a}$ (by putting $q = \frac{1}{r}$)

whereof the Fluent, when $x = a$, will consequently be the true Value required.

But the Fluent of the first Term ($\dot{x} q^x$) is found = $\frac{q^x}{m}$; and That of the Second $\left(\frac{x \dot{x} q^x}{a} \right) = \frac{q^x}{ma} \times x - \frac{1}{m}$;

by Prob. I, Sect. 6, Vol. 2, of my *Doctrine and Application of Fluxions*: Where m denotes the Hyperbolical Logarithm of q .

324 *Of the Values of Annuities and Reversions,*

These Fluents, corrected by their constant Quantities, become $\frac{q^x - 1}{m}$, and $\frac{q^x}{am} \times x - \frac{1}{m} + \frac{1}{am^2}$:

Which, when x is $= a$, are expressed by $\frac{q^a - 1}{m}$, and $\frac{q^a}{m} + \frac{1 - q^a}{am^2}$, respectively. The Difference of which, or $-\frac{1}{m} - \frac{1 - q^a}{am^2}$, is consequently the true Value of the proposed Life.

But $\frac{q^x - 1}{m}$ (the Fluent of $x q^x$), when x becomes infinite, will be equal to the Perpetuity, that is $-\frac{1}{m}$

$\left(= \frac{1}{\text{Hyp. Log. } r} \right) = p$: Whence, $-m$ being $= \frac{1}{p}$, the Value found above will, also, be expressed by $p - \frac{1 - q^a}{1 - q^a} \times \frac{p^2}{a}$, or, its Equal, $p - \frac{pM}{a}$; supposing M to be put for $(1 - q^a \times p)$ the Value, *certain*, for the Time a .

The Value here brought, it may be observed, exceeds $\left(p - \frac{p + 1 \times M}{a} \right)$ That given in p. 316, by $\frac{M}{a}$:

Which is owing to the Advantage of enjoying an Estate in Land (by an actual Possession to the last Moments of Life) preferable to an Annuity, where no Regard is had to the Time elapsed from the last Yearly-Payment to the Dropping of the proposed Life.

P R O B L E M

PROBLEM II.

To determine the Value of an Estate in Land, to continue during the joint Existence of any Number of assigned Lives, A, B, C, D, &c. according to the aforesaid Hypothesis,

Let the Complements of the proposed Lives A, B, C, D, &c. be denoted by $a, b, c, d, \&c.$ respectively: Then, supposing other Things to remain as in the preceding Problem, the Value of the Expectation on the

Moment \dot{x} will, here, be expressed by $\frac{a-x}{a} \times \frac{b-x}{b} \times \frac{c-x}{c}$

$\times \frac{d-x}{d} \&c. \times \frac{\dot{x}}{r^x}$, or its equal, $1 - \frac{x}{a} \times 1 - \frac{x}{b} \times 1 - \frac{x}{c}$

$\&c. q^x \dot{x}$. Which, if the Sum of all the Quantities

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \&c.$ be put $= a$; the Sum of all their

Rectangles $= \beta$; the Sum of all their Solids $= \gamma$, &c. will be reduced to $\dot{x} q^x - a x \dot{x} q^x + \beta x^2 \dot{x} q^x - \gamma x^3 \dot{x} q^x + \delta x^4 \dot{x} q^x, \&c.$

Whereof the Fluent (by the Problem above quoted) will be

found $= \frac{q^x}{m}$ into $1 - a \times x - \frac{1}{m} + \beta \times x^2 - \frac{2x}{m} + \frac{2}{m^2}$

$- \gamma \times x^3 - \frac{3x^2}{m} + \frac{6x}{m^2} - \frac{6}{m^3} + \&c.$ which, when x

is $= 0$, becomes $= \frac{1}{m} + \frac{a}{m^2} + \frac{2\beta}{m^3} + \frac{6\gamma}{m^4} + \frac{24\delta}{m^5} \&c.$

Hence the corrected Fluent (by subtracting this last Expression, and writing p , instead of, its Equal, —

$\frac{1}{m}$) will come out $p - a p^2 + 2\beta p^3 - 6\gamma p^4 + 24\delta p^5, \&c.$

$- q^x p$ into $1 - a \times x + p + \beta \times x^2 + 2xp + 2p^2 -$

$\gamma \times x^3 + 3x^2 p + 6xp^2 + 6p^3, \&c.$ which, by taking x equal

326 *Of the Values of Annuities and Reversions,*

equal to the Complement of the oldest Life, gives the true Value required.

When all the Lives are equal, the Conclusion will become much more simple: For, the Fluxion above given, by making $z = a - x$ ($= b - x = c - x$, &c.) will be here transformed to $-\frac{z^n \dot{z} r^z}{a^n r^a}$. Whose Fluent, supposing

m to denote the Hyperbolical Logarithm of r , will ap-

pear to be $-\frac{r^z}{ma^n r^a} \times z^n - \frac{n z^{n-1}}{m} + \frac{n \cdot n-1 \cdot z^{n-2}}{m^2}$ &c.

Which, when $x = 0$, or $z = a$, becomes $-\frac{1}{ma^n} \times$

$a^n - \frac{na^{n-1}}{m} + \frac{n \cdot n-1 \cdot a^{n-2}}{mm}$ &c. But when $x = a$,

or $z = 0$, it becomes $-\frac{1}{ma^n r^a}$ into $\pm \frac{1 \cdot 2 \cdot 3 \cdot 4 \dots n}{m^n}$:

The Difference of which Values, or its Equal,

$p - \frac{np^2}{a} + \frac{n \cdot n-1 \cdot p^3}{a^2} - \frac{n \cdot n-1 \cdot n-2 \cdot p^4}{a^3}$ &c.

$\pm \frac{1 \cdot 2 \cdot 3 \cdot 4 \dots n}{a^n r^a} \times p^{n+1}$, is consequently the true

Value of all the Joint Lives: where the Sign of the last Term must be negative, or positive, according as (n) the Number of Lives is even or odd.

P R O B L E M III.

Suppose that a given Sum (S) is to be received, as a Legacy, on the Decease of B, in case A is then extinct: To find its Value in present Money, allowing for that Contingency.

If a, b, q , &c. be supposed as in the preceding Problems, it is easy to perceive that the Expectation on the Moment \dot{x} , will here be defined by $\frac{x}{a} \times \frac{\dot{x}}{b} \times$
 Sq^x

Sq^x , or its Equal, $\frac{S}{a} \times \frac{x \dot{x} q^x}{b}$. But the Fluent of $\frac{x \dot{x} q^x}{b}$

(when $x = b$) is known to express the Difference between the Value of the Life B, and the Value (M) of an Annuity certain, for b Years (*Vid. Prob. 1*). Whence

it plainly follows, that $\frac{S}{a} \times \overline{M-B}$, will be the true Value sought; provided B is older than A.

But, if B be younger than A; then, besides the Expectation depending on the Chance of his dying within the Term (a) to the End of which A has a Possibility of living, there is a farther Expectation arising from the Probability of B's not dying till after the Expiration of the said Term. Which Expectation, with regard to any particular Moment x (as A is here out of the Question)

will be represented by $\frac{x}{b} \times Sq^x$. Whose Fluent (generated while x is increased from a to b) is expressed by

$\frac{S}{b}$ drawn into the present Value (V) of an Annuity certain, for the Time ($b-a$) which B has a Chance of living after the greatest Limit of A's Duration. This,

therefore, added to $\frac{S}{b} \times \overline{N-A}$ (found as above, by

changing a for b , A for B , and N for M) gives $\frac{S}{b} \times$

$\overline{N+V-A}$, for the whole Value of the Expectation, in this Case. But (N) the Value of an Annuity certain for the Time a , with (V) the Value of the Reversion Thereof for the Time $b-a$, after the Time a , is consequently equal to Its whole Value (M) for the Time

b . Therefore our last Expression is reduced to $\frac{S}{b} \times$

$\overline{M-A}$. From which and the other Value found above, we have the following General Rule.

From

From the Value of an Annuity in Land, *certain* for as many Years as are expressed by the Complement of B's Age, subtract the Value of the oldest Life (be it which it will) then say, as the Complement of the youngest Life, is to the Remainder, so is the proposed Sum to its required Value in present Money.

P R O B L E M IV.

Q and his Heirs are intitled to an Estate of a given Value (S) on the Decease of C, in Case B survives A, and is himself survived by C: To determine the present Value of Q's Expectation.

Here the Expectation on the Moment x depends on the Probability of C's dying in that Moment, and on the Probability that both A and B are before extinct, with the further Restriction of A's dropping first.

In order to determine the latter of These, it will be proper to observe, from the last Problem, that, as $\frac{x}{a} \frac{x}{b}$ denotes the Probability of the Expectant's receiving the Sum S in the Moment x , so $\left(\frac{x^2}{2ab}\right)$ the Fluent thereof must consequently express the Probability of receiving it during the Interval x ; which is, evidently, the Probability here required. Hence it follows that the Expectation on the Moment x , in the present Case, is truly defined by $\frac{x}{c} \times \frac{x^2}{2ab} Sq^x$. Whose Fluent, $\frac{Sq^x}{2abcm} \times$

$x^2 - \frac{2x}{m} + \frac{2}{mm}$, properly corrected, will consequently be the true Value of Q's Expectation.

After the same manner the Value sought, in more complicated Cases, may be determined. But I shall here put down a general Theorem for finding the Value of an Annuity, granted upon any Number (n) of assigned Lives, A, B, C, D, &c. but so, as to continue,

tinue, only, as long as a given Number (m) of Them are in Being.

Let Q be the Value of all the Joint Lives $A, B, C, \&c.$ that is, the Value of an Annuity, for as long as They all continue in Being together; also let R be the Sum of the Values of all the Joint Lives that can arise, by combining $A, B, C, \&c.$ so as to leave out one Life at each Combination; and S the Sum of all the Joint Lives that can arise by combining the same, so as to leave out two Lives at each Combination, $\&c. \&c.$

Then will the Value of the Purchase be truly expressed by $\pm \frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} (n-m) \times Q \mp \frac{n-2}{1} \times \frac{n-3}{2} \times \frac{n-4}{3} (n-m-1) \times R \pm \frac{n-3}{1} \times \frac{n-4}{2} \times \frac{n-5}{3} (n-m-2) \times S \mp \frac{n-4}{1} \times \frac{n-5}{2} \times \frac{n-6}{3} (n-m-3) \times T, \&c.$

Where the upper, or the lower, Signs obtain, according as $n-m$ is an even, or an odd, Number; and where the Quantities between the Hooks, express the Number of the preceding Factors to be taken; with regard to which it is to be observed, that in the last Term, where the Number of Factors becomes $= 0$, an Unit must be taken.

This Theorem (which is strictly true according to any Law of the Decrements of Life) is the same, in effect, with that given at *p.* 26, of my Doctrine of Annuities, but rather more commodious.—Nevertheless, as the Trouble of computing the Values $Q, R, S, \&c.$ will still be very great, when several Lives are concerned, the following Approximation for the Value of the longest Life (as the Annuity is usually held thereon) will also be found of Use.

Let $a, b, c, d, e, f, \&c.$ express the Values of any Number of single Lives; of which Values let a be the greatest, b the next, and so on; also let m denote the Interest and eight Tenths of the Interest of one Pound for one Year:

Z

Then

330 *Of the Values of Annuities and Reversions,*

Then will the Value of all the Lives be nearly $= a + \frac{2-mb \times b^2}{4a} + \frac{2-mc \times c^3}{9ab} + \frac{2-md \times d^4}{16abc} + \frac{2-me \times e^5}{25abcd} + \text{&c.}$

Which Theorem, in that Case where all the Lives are equal, will become much more simple, being then reducible to

$a + 2-ma \times a \times \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}, \text{ &c.}$ Where as many Terms of the Series $\frac{1}{4} + \frac{1}{9} + \frac{1}{16} \text{ &c.}$ are to be taken, as are expressed by the Number of the Lives, less one.

The E N D.

ERRATA.

Page 11, l. 11, for $4d$ read $2d$; p. 61, l. last, for g r. $\frac{1}{2}g$; p. 73, l. 23, after *either*, r. *be*; p. 81, l. 11, for a r. 77; p. 82, l. 18, for $= \overline{2mn}^2$ r. $+ \overline{2mn}^2$; p. 87, l. 19, for 51, r. 52; l. 27, for 26 and 27, r. 29 and 30; p. 89, l. 4, for BC r. AC; p. 98, l. 10, for b r. a , and for a r. b ; p. 99, l. 24, for HG r. HI; p. 106, l. 10, for EK r. IK; p. 109, l. 27, for mma , under the Radical Sign, r. mna ; p. 126, l. 18, r. *to, and from*; p. 133, l. 18, for *inscribed* r. *described*; p. 157, l. 20, for HKL r. KHL; p. 166, l. 16, for ADE r. AED; p. 174, l. 3, for HN r. HL; p. 184, l. 11, dele *with*; p. 180, l. 7, for *Power* r. *Powder*; p. 214, l. 12, for 3000 r. 6000; p. 228, l. 6, r. *Equations*; l. 20, for $\frac{n-1}{1}$ r. $\frac{n-1}{2}$; p. 231, l. 8, for $1-2x$ r. $4 \times 1-2x$; p. 234, l. 28, dele the *Comma*, and before *Fluxions* r. *the*; p. 311, l. 14, before *profuse* r. *a*.



